MA3203 - Problem Set 7 (Length/Jordan-Hölder)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise specified.

1. Let Λ be a ring, let M be a module of finite length, and let $N \subseteq M$ be a submodule.
   (a) Show that $M \cong N$ if and only if $\ell(M) = \ell(N)$.
   (b) Show that M/N is simple if and only if there does not exist a submodule L so that $N \subsetneq L \subsetneq M$.

2. Let A_∞ be the "quiver" with vertex set $(A_\infty)_0 = \mathbb{Z}$ and an arrow $\alpha_i : i \to i+1$ for each $i \in \mathbb{Z}$. Define a representation (V, f) so that $V(i) = K$ and $f_{\alpha_i} = 1_K$ for all i.
   (a) What are the subrepresentations of (V, f)?
   (b) Show that (V, f) does not have finite length.

3. Consider the quiver $Q = \ 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$. For $\lambda \in K$, define a representation $M(1, \lambda) =$ $K \overset{\lambda}{\underset{1}{\rightrightarrows}} K$. Likewise define $M(1, \infty) = \ K \overset{1}{\underset{0}{\rightrightarrows}} K$. We recall from a previous problem set that each of these representations is indecomposable.
   (a) For all $\Lambda, \Lambda' \in K \cup \{\infty\}$, show that
   $$\mathrm{Hom}_\Lambda(M(1,\lambda), M(1,\lambda')) \cong \begin{cases} K & \lambda = \lambda' \\ 0 & \lambda \neq \lambda', \end{cases}$$
   where $\mathrm{Hom}_\Lambda(-,-)$ is the (vector space of) morphisms of Λ-modules.
   (b) Let $\mathcal{R}$ be the extension closure of $\{M(1,\lambda) : \lambda \in K \cup \{\infty\}\}$. Modules in $\mathcal{R}$ are typically called *regular*. Part (a) implies that the representations in $\{M(1,\lambda) : \lambda \in K \cup \{\infty\}\}$ behave like simple modules when we consider only those modules which are regular. Because of this, these modules are sometimes called "quasi simple". (There is nothing to show for this part.)
   (c) Given $\lambda \in K$ and $n > 1$, let $B(n, \lambda)$ be the $n \times n$ matrix with λ on the diagonal and 1 above the diagonal. For example, $B(2,\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and $B(3,\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

Define a representation $M(n,\lambda) = K^n \underset{\mathrm{Id}}{\overset{B(n,\lambda)}{\rightrightarrows}} K^n$. Likewise, define a representation $M(n,\infty) = K^n \underset{B(n,0)}{\overset{\mathrm{Id}}{\rightrightarrows}} K^n$. Show that there is an exact sequence

$$0 \to M(n-1,\lambda) \to M(n,\lambda) \to M(1,\lambda) \to 0$$

for any $\lambda \in K \cup \{\infty\}$ and $n > 1$.

(d) Conclude by induction that each $M(n,\lambda)$ is regular.

(e) Each regular module comes equipped with a "quasi composition series", which is a filtration

$$M_0 = 0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

where each M_i is regular and each M_i/M_{i+1} is quasi simple. The value of n is then referred to as the "quasi length" of M. Show that the quasi length of $M(n,\lambda) = n$ for all $\lambda \in K \cup \{\infty\}$ and $n \geq 1$. *Hint: use part c.*

4. Let Λ be a ring and M, N modules. We say two exact sequences of the form

$$0 \longrightarrow N \xrightarrow{f_1} E_1 \xrightarrow{g_1} M \longrightarrow 0,$$

$$0 \longrightarrow N \xrightarrow{f_2} E_2 \xrightarrow{g_2} M \longrightarrow 0$$

are equivalent if there exists an isomorphism $\phi : E_1 \to E_2$, and so that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f_1} & E_1 & \xrightarrow{g_1} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle Id_N} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle Id_M} & & \downarrow \\ 0 & \longrightarrow & N & \xrightarrow{f_2} & E_2 & \xrightarrow{g_2} & M & \longrightarrow & 0. \end{array}$$

The set of such short exact sequences up to this equivalence relation is denoted by $\mathrm{Ext}^1_\Lambda(M,N)$.

Let $m \geq 0$ and suppose $\Lambda = KQ$, where $Q = 1 \underset{\alpha_m}{\overset{\alpha_1}{\underset{\vdots}{\rightrightarrows}}} 2$. Recall that S_i denotes the simple representation with $S_i(i) = K$ and $S_i(j) = 0$ for all other vertices. Show that $\mathrm{Ext}^1_\Lambda(S_1,S_2)$ has the structure of an m-dimensional vector space. *Hint: try the cases $m = 1$ and $m = 2$ first.*