MA3203 - Problem Set 5 (Length)

In all problems, K denotes a field and all representations are finite dimensional representations over K.

1. [1, Exercise 2.1cd] Find a composition series for the module Λe_1, where Λ is the path algebra of each of the following.

   (a) $\Lambda = KQ$ for $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

   (b) $\Lambda = KQ/(\beta\alpha)$ for $Q = \; 1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3$

   (c) $\Lambda = KQ$ for $Q = \; 1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3$

   (d) $\Lambda = KQ/(\beta\alpha, \delta^3)$ for $Q = \; 1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3 \circlearrowright \delta$

2. Let Q be the quiver $\; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \;$. Consider the representations of Q

$$M : K^2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} K \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} K^2, \qquad N : K \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K^2$$

   (a) Find a composition series for each of M and N.

   (b) What do you notice about the number of times each composition factor appears?

   (c) (Challenge) How many different composition series does each of M and N have. *Hint: start by looking for subrepresentations of one smaller dimension.*

3. A representation (or module) is called *uniserial* if has exactly one composition series.

   (a) Let $\Gamma = 1 \to 2 \to 3 \leftarrow 4$. Show that the representation $K \xrightarrow{1} K \xrightarrow{1} K \leftarrow 0$ of Γ is uniserial.

   (b) Again let Let $\Gamma = 1 \to 2 \to 3 \leftarrow 4$. Show that the representation $0 \to K \xrightarrow{1} K \xleftarrow{1} K$ of Γ is not uniserial.

   (c) Let M be a representation of an arbitrary quiver with relations (Q, ρ). Show that if M is uniserial then every subrepresentation of M which appears in its unique composition series[1] is also uniserial.

[1]We will prove next time that every subrepresentation of a finite-length representation appears in some composition series. In particular, this will imply that the unique composition series of a uniserial representation contains all of its subrepresentations.

(d) Let M and N be (left) modules over some ring Λ. Show that if M and N have finite length, then $M \oplus N$ has finite length and is not uniserial.

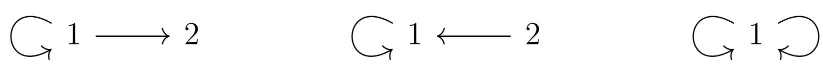
4. Let (Q, ρ) be a quiver with relations for which $(\rho) \subseteq KQ$ is an admissible ideal (recall that this means $(Q_1)^m \subseteq (\rho) \subseteq (Q_1)^2$ for some $m > 1$). The bound quiver algebra $KQ/(\rho)$ is called a *Nakayama algebra* if every indecomposable representation of (Q, ρ) is uniserial.

(a) Show that if Q is one of:

$$1 \leftarrow 2 \rightarrow 3, \qquad 1 \rightarrow 2 \leftarrow 3, \qquad 1 \rightrightarrows 2$$

then $KQ/(\rho)$ is not a Nakayama algebra. *Hint: For each of these quivers, the only admissible ideal is the 0 ideal, and so there are no relations. Now look for examples similar to those you found in problem 3.*

(b) Show that if Q is one of

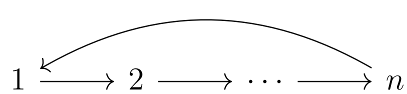


then $KQ/(\rho)$ is not a Nakayama algebra. *Hint: The first quiver looks identical to the first quiver in part (a), but with the first two vertices glued together.*

(c) Conclude that if Q contains a vertex which is the source of at least two arrows or the target of at least two arrows, then $KQ/(\rho)$ is not a Nakayama algebra.

(d) It turns out that the converse of (c) is true as well, but we won't prove this. In particular, if Q is connected, this means $KQ/(\rho)$ is Nakayama if and only if Q is one of:

$$A_n = \ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n,$$

$\Delta_n =$ 

for some $n \geq 1$. (Again, part (d) has nothing to solve/prove.)

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.