MA3203 - Problem Set 4 (Quivers with Relations)

In all problems, K denotes a field and all representations are representations over K.

1. (Based on [1, Exercise II.14]) Let $Q = \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$ and let $\Lambda = KQ/(\alpha^2, \beta^2, \beta\alpha)$.

   (a) What is the dimension of Λ as a vector space?

   (b) Let $K\langle x, y\rangle$ be the ring of polynomials with coefficients in K and noncommuting variables x and y (that is, $xy \neq yx$). Show that $\Lambda \cong K\langle x, y\rangle/(x^2, y^2, yx)$.

   (c) Let $A = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & a \end{bmatrix} : a, b, c, d \in K \right\}$. Show that A is a ring and that $\Lambda \cong A$.

2. Let $Q = (Q_0, Q_1)$ be an arbitrary quiver. The ideal of KQ generated by Q_1 is called the *arrow ideal* of Q (or technically KQ), and is often denoted R_Q. An arbitrary ideal $I \subseteq KQ$ is called *admissible* if there exists an integer $m \geq 2$ so that $R_Q^m \subseteq I \subseteq R_Q^2$.

   (a) Show that if I is admissible, then KQ/I is finite dimensional (even if KQ is not).

   (b) [1, Exercise II.7a] Let $\Gamma = \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\gamma"] \end{tikzcd}$ and let $\rho = \{\alpha^2, \beta\gamma, \gamma\beta - \gamma\alpha\beta\}$. Show that (ρ) is an admissible ideal.

   (c) Compute $\dim_K K\Gamma/(\rho)$.

3. Let $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\alpha^*"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\beta^*"] \end{tikzcd}$ and let $\Lambda = KQ/(\alpha^*\alpha, \beta\beta^*, \alpha\alpha^* + \beta^*\beta)$. This algebra is called the *preprojective algebra of type* A_3 and has many interesting properties.

   (a) Find the representation corresponding to the module Λe_i for $i = 1, 2, 3$.

   (b) Show that there exist morphisms $\Lambda e_2 \to \Lambda e_2$ which are not invertible.

   (c) Show that Λe_2 is indecomposable.

   (d) Find the representation corresponding to Λ considered as a module over itself. *Hint: recall that $\Lambda = \Lambda \cdot 1_\Lambda$. What is the identity element of Λ?*

4. [1, Exercise II.15] Find a bound quiver algebra which is isomorphic to each of the following triangular matrix algebras:

$$\text{(a)} \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & K & K & 0 \\ K & K & K & K & K \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ K & K & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}$$

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).