MA3203 - Problem Set 2 (Representations of Quivers)

In all problems, K denotes a field and all representations are representations over K.

1. [1, Exercise 1.1] Let Γ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

   (a) Consider the representations of Γ

   $$(V,f) : K^2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix}} K \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} K^2, \qquad (V',f') : K \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} K^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}} K^2$$

   and describe all morphisms $(V,f) \to (V',f')$ and $(V',f') \to (V,f)$.

   (b) Given two representations (W,g) and (W',g') of an arbitrary quiver Q, we say that (W',g') is a *subrepresentation* of (W,g) if $W'(i)$ is a subspace of $W(i)$ for every vertex $i \in Q_0$ and $g'_\alpha = g_\alpha|_{W(i)}$ for every arrow $(\alpha : i \to j) \in Q_1$. Find all subrepresentations of the representation $K \xrightarrow{1} K \xrightarrow{1} K$ of the quiver Γ from part (a).

   (c) (Challenge) Give a definition of a *factor representation* and find all factor representations of the representation $K \xrightarrow{1} K \xrightarrow{1} K$ of the quiver Γ from part (a).

2. Let $\Gamma = \; 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$. For all $\lambda, \mu \in K \setminus \{0\}$, define a representation $M(\lambda,\mu)$ of Γ by

   $$M(\lambda,\mu) = \; K \overset{\mu}{\underset{\lambda}{\rightrightarrows}} K \; .$$

   (a) Show that $M(\lambda,\mu) \cong M(\lambda',\mu')$ if and only if $\lambda'\lambda^{-1} = \mu'\mu^{-1}$.

   (b) Let $\mathcal{M}$ be the set of isomorphism classes of representations of the form $M(\lambda,\mu)$; that is, if $M(\lambda,\mu) \cong M(\lambda',\mu')$, then these representations correspond to the same element of $\mathcal{M}$. Show that there is a 1-1 correspondence between M and $K \setminus \{0\}$ given by $M(\lambda,\mu) \mapsto \lambda\mu^{-1}$, and construct its inverse.

   (c) Let $n \in \mathbb{N}$. For all pairs of invertible matrices $L, N \in M_n(K)$, define a representation

   $$M(L,N) = \; K^n \overset{L}{\underset{N}{\rightrightarrows}} K^n$$

   of Γ. Show that $M(L,N) \cong M(L',N')$ if and only if $L^{-1}N$ and $(L')^{-1}N'$ are similar matrices.

3. Let $\Gamma = {}_{\alpha}\circlearrowleft 1 \circlearrowright_{\beta}$ and let $n \in \mathbb{N}$, and suppose that $K = \mathbb{C}$. For all pairs of invertible matrices $L, N \in M_n(\mathbb{C})$, define a representation

$$M(L,N) = {}_{L}\circlearrowleft \mathbb{C}^n \circlearrowright_{N}$$

of Γ. Show that if $M(L,N)$ and $M(L',N')$ are isomorphic, then L is similar to L' and N is similar to N', but that the converse is not true in general.

4. Let Γ be an arbitrary quiver and let $\phi : (V,f) \to (V',f')$ be a morphism of representations of Γ. We define the *kernel* of ϕ to be the representation $\ker(\phi) = (W,g)$ given by

$$\begin{aligned} W(j) &= \ker(\phi_j) \qquad && j \in \Gamma_0 \\ g(\alpha) &= f_\alpha|_{\ker(\phi_i)} \qquad && (\alpha : i \to j) \in \Gamma_1 \end{aligned}$$

   (a) Show that $\ker(\phi)$ is well-defined; that is, that $\operatorname{Im}(f_\alpha|_{\ker(\phi_i)}) \subseteq \ker(\phi_j)$ for all arrows $\alpha : i \to j$.

   (b) Find the kernel of some of the morphisms you found in problem 1.

   (c) Give a definition of the *image* of a morphism of representations and find the image of some of the morphisms you found in problem 1.

   (d) (Challenge) Show that for any morphism of representations $\phi : (V,f) \to (V',f')$ there is a surjective morphism $(V,f) \twoheadrightarrow \operatorname{Im}\phi$ with kernel $\ker(\phi)$; that is, that $(V,f)/\ker\phi \cong \operatorname{Im}\phi$.

5. Let $\Gamma = 1 \circlearrowright_{\alpha}$. For $\lambda \in K$, define a representation $M(\lambda) = K \circlearrowright_{\lambda}$ of Γ. Likewise, for $n \in \mathbb{N}$ and any matrix $N \in M_n(K)$, define a representation $M(N) = K^n \circlearrowright_{N}$.

   (a) Show that $M(\lambda)$ is simple (its only subrepresentations are the 0 representation and itself).

   (b) Show that there is a nonzero morphism $M(\lambda) \to M(N)$ if and only if λ is an eigenvalue of N.

   (c) Show that $M(\lambda)$ and $M(\lambda')$ are only isomorphic if $\lambda = \lambda'$.

   (d) Conclude that if K is algebraically closed, then $\{M(\lambda) : \lambda \in K\}$ is the set of all finite dimensional simple representations of Γ up to isomorphism.

   (e) Suppose that $K = \mathbb{R}$. Find a representation M of Γ which is simple and which has $\dim_{\mathbb{R}} M > 1$.

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.