MA3203 - Problem Set 1 Answers and Hints

1a) Let $B_{i,j}$ denote the 3×3 matrix which has 1 in position (i,j) and 0 elsewhere. $K\Gamma$ is isomorphic to the algebra $\begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix}$ via the map sending each e_i to $B_{i,i}$, sending α to $B_{2,1}$, sending β to $B_{3,2}$. Note that we extend multiplicatively (since an isomorphism of K-algebras is in particular an isomorphism of rings), so that $\beta\alpha$ is sent to $B_{3,2}B_{2,1} = B_{3,1}$.

1b) Let $\Lambda = \begin{bmatrix} K & 0 & 0 \\ K^2 & K & 0 \\ K^2 & K & K \end{bmatrix}$ with multiplication

$$\begin{bmatrix} a & 0 & 0 \\ (b,c) & d & 0 \\ (e,f) & g & h \end{bmatrix} \begin{bmatrix} a' & 0 & 0 \\ (b',c') & d' & 0 \\ (e',f') & g' & h' \end{bmatrix} = \begin{bmatrix} aa' & 0 & 0 \\ (a'b+db', a'c+dc') & dd' & 0 \\ (a'e+gb'+he', a'f+gc'+hf') & gd'+hg' & hh' \end{bmatrix}.$$

The $K\Gamma$ is isomorphic to Λ via the map sending each e_i to $B_{i,i}$, sending γ to $M_{3,2}$, sending α to the matrix with $(1,0)$ in position $(2,1)$ and 0 elsewhere, and sending β to the matrix with $(0,1)$ in position $(2,1)$ and 0 elsewhere.

2a) Recall that a ring is local if and only if the set of non-invertible elements forms a (2-sided) ideal. If Γ contains an arrow α, then neither α nor $1-\alpha$ is invertible in $K\Gamma$, meaning $K\Gamma$ is not local. Otherwise, $K\Gamma$ is semisimple and thus only local if $|\Gamma_0| = 1$.

4a) We can define an isomorphism $\Phi : K\Gamma \to K\Gamma$ by

$$\begin{aligned} \Phi(e_i) &= e_i, \qquad && i \in \Gamma_0 \\ \Phi(\rho) &= \rho, \qquad && \rho \in \Gamma_1 \setminus \{\delta\} \\ \Phi(\delta) &= -\delta \end{aligned}$$

and extending multiplicatively and additively. We note that Φ sends the ideal $(\beta\alpha + \delta\gamma)$ to the ideal $(\beta\alpha - \delta\gamma)$ and thus induces the desired isomorphism between the quotient algebras. Now, since both ideals are 1-dimensional as vector spaces, they are equal if and only if $1 = -1$ in K (and hence K has characteristic 2).