

MA3203 - Problem Set 1 (Path Algebras)

In all problems, K denotes a field and all ideals are left ideals.

1. (based on [1, Exercise II.3]) Construct an isomorphism of K -algebras from the path algebra of each of the following quivers to a triangular matrix algebra.

(a) $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 .$

(b) $1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 \xrightarrow{\gamma} 3 .$

(c) $1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\gamma} \\ \xrightarrow{\beta} \end{array} 2 \xrightarrow{\beta} 3 .$

2. [1, Exercise II.2cd] Let Γ be a connected quiver; that is, a quiver for which there exists an *undirected* path between any two vertices.

(a) Show that $K\Gamma$ is a local ring if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| = 0$.

(b) Show that $K\Gamma$ is commutative if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| \leq 1$.

(c) Are there quivers which are not connected for which $K\Gamma$ is local or commutative?

3. [2, Exercise 1.2abc] Let $\Lambda = \begin{bmatrix} K & 0 \\ K[x]/(x^2) & K \end{bmatrix}$. Note that Λ can be seen as a subset of the ring of 2×2 matrices over $K[x]/(x^2)$.

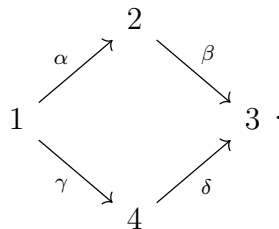
(a) Show that Λ is a ring. How can Λ be made into a K algebra?

(b) Let $I = \begin{bmatrix} 0 & 0 \\ K[x]/(x^2) & 0 \end{bmatrix}$. Show that I is an ideal of Λ and $\Lambda/I \cong K \oplus K$ (as rings).

(c) Let $\Gamma = 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 .$ Construct an isomorphism of K -algebras $K\Gamma \cong \Lambda$.

4. (based on [1, Exercise II.17])

(a) Let Γ be the quiver



Show that $K\Gamma/\langle\beta\alpha + \delta\gamma\rangle \cong K\Gamma/\langle\beta\alpha - \delta\gamma\rangle$, but (assuming $\text{char}K \neq 2$) the two ideals are not equal.

(b) Let Γ be the quiver $1 \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} 2 \xrightarrow{\gamma} 3$. Show that $K\Gamma/\langle\gamma\beta\rangle \cong K\Gamma/\langle\gamma\beta - \gamma\alpha\rangle$, but the two ideals are not equal.

5. Let $\Gamma = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\Gamma' = 1 \begin{matrix} \xrightarrow{\gamma} \\ \xrightarrow{\alpha} \end{matrix} 2 \xrightarrow{\beta} 3$. The ideal $\langle\gamma - \beta\alpha\rangle$ is not what is called an “admissible ideal” since it involves a path of length 1. Show that $K\Gamma'/\langle\gamma - \beta\alpha\rangle \cong K\Gamma$.

References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).
- [2] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, <http://wiki.math.ntnu.no/ma3203/2017v/ovinger>.