MA3203 - Problem Set 1 (Path Algebras)

In all problems, K denotes a field and all ideals are left ideals.

1. (based on [1, Exercise II.3]) Construct an isomorphism of K-algebras from the path algebra of each of the following quivers to a triangular matrix algebra.

   (a) $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

   (b) $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \end{tikzcd}$.

   (c) $\begin{tikzcd} 1 \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & 2 \arrow[r, "\beta"'] & 3 \end{tikzcd}$.

2. [1, Exercise II.2cd] Let Γ be a connected quiver; that is, a quiver for which there exists an *undirected* path between any two vertices.

   (a) Show that $K\Gamma$ is a local ring if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| = 0$.

   (b) Show that $K\Gamma$ is commutative if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| \leq 1$.

   (c) Are there quivers which are not connected for which $K\Gamma$ is local or commutative?

3. [2, Exercise 1.2abc] Let $\Lambda = \begin{bmatrix} K & 0 \\ K[x]/(x^2) & K \end{bmatrix}$. Note that Λ can be seen as a subset of the ring of 2×2 matrices over $K[x]/(x^2)$.

   (a) Show that Λ is a ring. How can Λ be made into a K algebra?

   (b) Let $I = \begin{bmatrix} 0 & 0 \\ K[x]/(x^2) & 0 \end{bmatrix}$. Show that I is an ideal of Λ and $\Lambda/I \cong K \oplus K$ (as rings).

   (c) Let $\Gamma = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$. Construct an isomorphism of K-algebras $K\Gamma \cong \Lambda$.

4. (based on [1, Exercise II.17])

   (a) Let Γ be the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 3 \\ & 4 \arrow[ur, "\delta"'] & \end{tikzcd} .$$

   Show that $K\Gamma/\langle \beta\alpha + \delta\gamma \rangle \cong K\Gamma/\langle \beta\alpha - \delta\gamma \rangle$, but (assuming $\mathsf{char} K \neq 2$) the two ideals are not equal.

   (b) Let Γ be the quiver $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \end{tikzcd}$. Show that $K\Gamma/\langle \gamma\beta \rangle \cong K\Gamma/\langle \gamma\beta - \gamma\alpha \rangle$, but the two ideals are not equal.

5. Let $\Gamma = \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \end{tikzcd}$ and $\Gamma' = \begin{tikzcd} 1 \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & 2 \arrow[r, "\beta"'] & 3 \end{tikzcd}$. The ideal $\langle \gamma - \beta\alpha \rangle$ is not what is called an "admissible ideal" since it involves a path of length 1. Show that $K\Gamma'/\langle \gamma - \beta\alpha \rangle \cong K\Gamma$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.