MA3203 - Problem Set 18 (Injectives)

Let K be a field.

1. [1, Exercise 5.2] Let $Q = \ 1 \longrightarrow 2 \longrightarrow 3$ and let $\Lambda = KQ$.

   (a) Find the indecomposable finitely-generated injective Λ-modules.

   (b) Find the socles and injective envelopes of the following representations of Q:

      i. $K \xrightarrow{0} K \xrightarrow{0} K$

      ii. $K \xrightarrow{1} K \xrightarrow{0} K$

      iii. $K \xrightarrow{[1\ 0]} K^2 \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} K$

2. [1, Exercise 5.3] Let $Q = \ 1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3$, let $\rho = \{\beta\alpha\}$, and let $\Lambda = KQ/(\rho)$. Find the socles and injective envelopes of the following representations of (Q, ρ):

   (a) $K \xrightarrow{1} K \overset{0}{\underset{\begin{bmatrix}0\\1\end{bmatrix}}{\rightrightarrows}} K^2$

   (b) $K \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} K^2 \overset{[1\ 0]}{\underset{[1\ 1]}{\rightrightarrows}} K$

   (c) $0 \longrightarrow K^2 \overset{\begin{bmatrix}1&0\\1&0\end{bmatrix}}{\underset{\begin{bmatrix}1&0\\1&0\end{bmatrix}}{\rightrightarrows}} K^2$

3. Let $\Lambda = KQ/I$ for some quiver Q and admissible ideal I. For $i \in |Q_0|$, let I_i be the injective (left) module corresponding to the projective Λe_i (so that there is an injective envelope $\Lambda e_i/\mathrm{rad}\Lambda e_i \to I_i$). Let M be a finitely generated left Λ-module.

   (a) Show that $\mathrm{Hom}_\Lambda(\Lambda e_i, M) \cong \mathrm{Hom}_\Lambda(M, I_i)$.

   (b) Show that $\mathrm{Hom}_\Lambda(M, I_i) \neq 0$ if and only if the socle $\mathrm{soc}(I_i)$ is a composition factor of M.

4. Let Λ be a ring and let I be an injective left Λ-module. Suppose I has a projective cover $P \xrightarrow{f} I$.

(a) Let $P \xrightarrow{g} J$ be the injective envelope of P. Show that there is a surjection $J \xrightarrow{h} I$ so that $h \circ g = f$.

(b) Suppose J has a projective cover $Q \xrightarrow{i} J$. Show that there exists P' (possibly $P' = 0$) so that $Q \cong P \oplus P'$.

(c) (Challenge) Let Q be the quiver $1 \leftarrow 2 \rightarrow 3$ and let $\Lambda = KQ$. Show that the simple module S_2 is injective, and that P' as defined in (b) is nonzero.

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.