MA3203 - Problem Set 17 Hints and Answers

(1a) Let $0 \neq x \in V$ and let B be a basis containing x. Let $x^* \in D(V)$ be the element of the dual basis B^* so that $x^*(x) = 1$. Then $\varphi_V(x)(x^*) = 1 \neq 0$. This means $x \notin \ker \varphi_V$.

(1b) Let $f : V \to W$. We need to show that $DD(f) \circ \phi_V = \phi_W \circ f$. These are both maps $V \to DD(W)$, so we need to show that $DD(f) \circ \phi_V(x) = \phi_W \circ f(x)$ for all $x \in V$. These in turn are both elements of $DD(V) = \mathrm{Hom}_K(D(V), K)$, so let $g \in D(V)$. We then have

$$\begin{array}{rcl} [DD(f) \circ \varphi_V(x)](g) & = & \varphi_V(x)(g \circ f) \\ & = & g \circ f(x) \\ & = & g(f(x)) \\ & = & \varphi_W(f(x))(g) \\ & = & [\varphi_W \circ f(x)](g) \end{array}$$

The fact that each φ_V is an isomorphism then comes from (a) and the fact that $\dim_K(V) = \dim_K(DD(V))$.

(3a) Note that A, B, and C are all finite dimensional vector spaces over K. Now choose a vector space isomorphism $h : B \to A \oplus C$. Then we have the following commutative diagram, with both rows exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_C} & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{pr_C} & C & \longrightarrow & 0 \end{array}$$

The map ι_A is the inclusion map and the map pr_C is the projection map. Now applying D to this diagram, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D(C) & \xrightarrow{D(g)} & D(B) & \xrightarrow{D(f)} & D(A) & \longrightarrow & 0 \\ \uparrow & & {\scriptstyle 1_{D(C)}}\uparrow & & {\scriptstyle D(h)}\uparrow & & {\scriptstyle 1_{D(A)}}\uparrow & & \uparrow \\ 0 & \longrightarrow & D(C) & \xrightarrow{\iota_{D(C)}} & D(C) \oplus D(A) & \xrightarrow{pr_{D(A)}} & D(A) & \longrightarrow & 0. \end{array}$$

We know the bottom row is a short exact sequence and all of the vertical maps are isomorphisms. This can be used to show that the top row is a short exact sequence as well. The reverse implication follows from the fact that $DD(-) \cong \mathrm{Id}_{\mathrm{vec}(K)}$ as functors.

(3b) Let $0 \neq C \subseteq D(S)$ be a submodule of $D(S)$. Then by (1), $D(D(S)/C) \subsetneq S$ is a submodule of S. Since S is simple, we then have $D(S) = C$ and so $D(S)$ is simple.

(3c) Use induction on length. The base case is (3b) and the induction step follows from (3a).