

MA3203 - Problem Set 17 (Duality)

1. (Exercise 13.2 in the videos) Let K be a field and let $\text{vec}(K)$ be the category of finite-dimensional vector spaces over K . Given a finite-dimensional K -vector space V , consider $\varphi_V : V \rightarrow DD(V)$ so that for $x \in V$ and $f \in D(V)$, we have $\varphi_V(x)(f) = f(x)$.

(a) Show that φ_V is injective.

(b) Show that $\varphi = (\varphi_V)_{V \in \text{vec}(K)}$ gives an isomorphism of functors $\text{Id}_{\text{vec}(K)} \rightarrow DD(-)$.

2. (Exercise 13.3 in the videos) Let K be a field and let $f : V \rightarrow W$ be a morphism of finite-dimensional K -vector spaces. Let B and B' be bases for V and W , respectively, and suppose the corresponding matrix form of f is $m_B^{B'}(f) =: A$. Show that $D(f) : D(W) \rightarrow D(V)$ has matrix form $m_{(B')^*}^{B^*} = A^\top$.

3. (Lemma 53 in the videos) Let Λ be a finite-dimensional algebra over some field K .

(a) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be morphisms of finitely generated left Λ -modules. Show that the sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is exact if and only if the sequence

$$0 \rightarrow D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \rightarrow 0$$

is exact.

(b) Let S be a finitely generated left Λ -module. Show that S is simple if and only if $D(S)$ is simple (as a module over Λ^{op}).

(c) Let A be a finitely generated left Λ -module. Show that $\ell(A) = \ell(D(A))$.

4. (Challenge) [1, Exercise III.3] Let K be a field. Let $(Q, \{\rho\})$ be a quiver with relations so that (ρ) is an admissible ideal of KQ . Let Q^{op} be the opposite quiver of Q . This quiver has the same vertex set as Q and for each $i \xrightarrow{\alpha} j$ an arrow of Q there is an arrow $i \xleftarrow{\alpha^*} j$ of Q^{op} . Now let $\{\rho^{op}\}$ be so that a linear combination of paths $\sum_i c^i \alpha_1^i \cdots \alpha_{m_i}^i \in \{\rho\}$ if and only if $\sum_i c^i (\alpha^*)_{m_i}^i \cdots (\alpha^*)_1^i \in \{\rho^{op}\}$.

Now consider the equivalences of categories $G : \text{Rep}(Q, \{\rho\}) \rightarrow \text{mod}KQ/(\rho)$ and $F : \text{mod}KQ^{op}/(\rho^{op}) \rightarrow \text{Rep}(Q^{op}, \{\rho^{op}\})$. This gives a duality $F \circ D \circ G : \text{Rep}(Q, \{\rho\}) \rightarrow \text{Rep}(Q^{op}, \{\rho^{op}\})$.

- (a) Let (V, f) be a representation of $(Q, \{\rho\})$. Show that $F \circ D \circ G(V, f) = (DV, Df)$, where for each vertex i in Q^{op} , $DV_i := D(V_i)$ and for each arrow α^* in Q^{op} , $Df_{\alpha^*} := D(f_\alpha)$.
- (b) Let $\varphi : (V, f) \rightarrow (W, g)$ be a morphism in $\text{Rep}(Q, \{\rho\})$. Describe the morphism $F \circ D \circ G(\varphi)$.

References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).