MA3203 - Problem Set 16 (Basic algebras)

Let Λ be a finite-dimensional K-algebra over some field K. Recall that if Λ is basic, then as rings, $\Lambda/\text{rad}\Lambda \cong D_1 \oplus \cdots \oplus D_n$, where each D_i is a division algebra over K. The converse of this fact is true as well, and while not explicitly stated, is essentially proven in the videos.

1. This exercise shows that the assumption that K is algebraically closed is necessary in order to claim that every basic K-algebra is isomorphic to a quiver algebra.

   (a) Show that $\mathbb{C}$ is a basic $\mathbb{R}$-algebra.

   (b) Show that there is no quiver Q and admissible ideal I so that $\mathbb{C} \cong \mathbb{R}Q/I$.

2. For simplicity, let K be an algebraically closed field. Let $\phi : \Gamma \to \Lambda$ be a non-unital morphism of finite-dimensional K-algebras. This means we are not assuming $\phi(1_\Gamma) = 1_\Lambda$.

   (a) Let $I \subseteq \Lambda$ be a left ideal. Show that there is an injective homomorphism of K-vector spaces $\phi^* : \Gamma/\phi^{-1}(I) \to \Lambda/I$ given by $\phi^*(x + \phi^{-1}(I)) = \phi(x) + I$.

   (b) If I is a 2-sided ideal, show that ϕ is a ring homomorphism (which does not necessarily send 1 to 1).

   (c) From now on, suppose that Λ is basic. Show that if I is maximal in Λ, then either $\phi^{-1}(I)$ is maximal in Γ or $\phi^{-1}(I) = \Gamma$. *Hint: what is* $\dim_K(\Lambda/I)$*?*

   (d) Show that $\phi(\text{rad}\Gamma) \subseteq \text{rad}\Lambda$. *Hint: recall that the radical is the intersection of all maximal left ideals.*

   (e) From now on, assume that ϕ is injective. Show that $\phi^{-1}(\text{rad}\Lambda) \subseteq \text{rad}\Gamma$. *Hint: recall that the radical is nilpotent and that any nilpotent ideal is contained in the radical.*

   (f) (Recall that we are assuming ϕ is injective.) Show that $\text{rad}\Gamma = \phi^{-1}(\text{rad}\Lambda)$.

   (g) [1, Exercise III.2b] Recall that since Λ is basic, we have an isomorphism of rings $\Lambda/\text{rad}\Lambda \cong K^n$ for some n. Use this fact and parts (b) and (f) to show that Γ is basic (assuming that ϕ is injective).

3. [1, Exercise III.3] For simplicity, let K be an algebraically closed field. Let

$$Q = \alpha \circlearrowright 1 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 2 \circlearrowleft \beta$$

   and let $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$.

(a) Show that $KQ/(\rho)$ has a K-basis $\{e_1, e_2, \alpha, \beta, \gamma, \delta, \alpha^2, \beta^2, \delta\beta, \gamma\alpha, \gamma\alpha^2, \alpha^2\delta\}$.

(b) Show that the subalgebra of $KQ/(\rho)$ generated by $\{\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2\}$ is isomorphic to the algebra of 2×2 matrices over K.

(c) Conclude that $KQ/(\rho)$ is not basic. *Hint: Use problem 2.*

(d) Conclude that (ρ) is not an admissible ideal.

4. [2, Problem 6.1] Let K be a field and let $\phi : \Lambda \to \Gamma$ be a surjective homomorphism of finite-dimensional K-algebras.

(a) Let M be a left Γ-module. For $\lambda \in \Lambda$ and $m \in M$, define $\lambda.m := \phi(\lambda).m$. Show that this makes M into a left Λ-module.

(b) Let $M, N \in \text{Mod}\Gamma$. Show that $\text{Hom}_\Gamma(M, N) = \text{Hom}_\Lambda(M, N)$ (as subspaces of $\text{Hom}_K(M, N)$).

(c) Show that $M \cong N$ as Γ-modules if and only if $M \cong N$ as Λ-modules.

(d) Define an exact, full, and faithful functor $F : \text{mod}\Gamma \to \text{mod}\Lambda$.

References

[1] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press (1995).

[2] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.