MA3203 - Problem Set 15 Answers and Hints

(1a) Let $f, g \in \text{End}_\Lambda(A)^{op}$ and lett $a \in A$. Then

$$a.(fg) = (fg)(a) = g \circ f(a) = f(a).g = (a.f).g.$$

Moreover, we have that Id_A is the identity of $\text{End}_\Lambda(A)^{op}$ and $a.Id_A = a$ for all $a \in A$. It then remains to show that addition (in A or $\text{End}_\Lambda(A)^{op}$) distributes over this operation.

(1c) Let $g, h \in \gamma$ and let $f \in \text{Hom}_\Lambda(A, X)$. Then for $a \in A$, we have

$$((gh).f)(a) = f \circ (gh)(a) = f \circ h \circ g(a) = (g.(f \circ h))(a) = (g.(h.f))(a).$$

Moreover, we have that $Id_A.f = f \circ Id_A = f$ for all $f \in \text{Hom}_\Lambda(A, X)$. It then remains to show that addition (in $\text{Hom}_\Lambda(A, X)$ or Γ) distributes over this operation.

(2) This is just a note that if Λ is not commutative, then $\text{Hom}_\Lambda(A, B)$ is not necessarily a Λ-module. Let $f \in \text{Hom}_\Lambda(A, B)$ and let $x, y \in \Lambda$. The natural way to define a module structure would be to define $(x.f)(a) = x.(f(a))$ for $a \in A$. However, with this definition $x.f$ may not be a module homomorphism. Indeed, let $x, y \in \Lambda$, then for $a \in A$, we have

$$\begin{aligned} (x.f)(y.a) &= x.(f(y.a)) = x.(y.(f(a))) = (xy).(f(a)) \\ y.(x.f)(a) &= y.(x.(f(a))) = (yx).(f(a)), \end{aligned}$$

and these may not be equal.

(3a) Let $a \in A$. Then $t_2 \circ t_1(a) = 0$ since the top row is exact. This means $s_2 \circ g \circ t_1 = 0$, and so $g \circ t_1(a) \in \ker(s_2) = \text{image}(s_1)$. Since s_1 is injective, there then exists a unique $a' \in A'$ so that $g \circ t_1(a) = s_1(a')$. Define $f(a) := a'$.

To see that f is a morphism of Λ-modules, let $a, b \in A$ and $\lambda \in \Lambda$. It follows that

$$\begin{aligned} s_1 \circ f(\lambda a + b) &= g \circ t_1(\lambda a + b) \\ &= \lambda \circ g \circ t_1(a) + g \circ t_1(b) \\ &= \lambda s_1 \circ f(a) + s_1 \circ f(b) \\ &= s_1(\lambda f(a) + f(b)). \end{aligned}$$

Since s_1 is injective, this means $f(\lambda a + b) = \lambda f(a) + f(b)$.

Finally, suppose that g and h are isomorphisms. Then by replacing g and h with g^{-1} and h^{-1}, the same argument shows that there exists $f' : A' \to A$ so that $g^{-1} \circ s_1 = t_1 \circ f'$. Now for $a \in A$, we have

$$\begin{aligned} t_1 \circ f' \circ f(a) &= g^{-1} \circ s_1 \circ f(a) \\ &= g^{-1} \circ g \circ t_1(a) \\ &= t_1(a). \end{aligned}$$

Since t_1 is injective, this means $f' \circ f(a) = a$; that is, that $f' \circ f = Id_A$. The argument that $f \circ f' = Id_{A'}$ is analogous. We conclude that f' is the inverse of f, and so f is an isomorphism.