MA3203 - Problem Set 15 (Projectivization)

1. (From the introduction in the videos) Let Λ be an artin R-algebra and let $A \in \text{mod}\Lambda$. Let $\Gamma = \text{End}_\Lambda(A)^{op}$.

   (a) For $a \in A$ and $f \in \text{End}_\Lambda(A)^{op}$, define $a.f := f(a)$. Show that this makes A into a right Γ-module.

   (b) Show that $A = {}_\Lambda A_\Gamma$ is a Λ-Γ-bimodule; that is, $(\lambda.a).f = \lambda.(a.f)$ for all $\lambda \in \Lambda$, $a \in A$, and $f \in \Gamma$.

   (c) Let $X \in \text{mod}\Lambda$. For $g \in \Gamma$ and $f \in \text{Hom}_\Lambda(A, X)$, define $g.f \in \text{Hom}_\Lambda(A, X)$ so that $(g.f)(a) := f(g(a))$ for all $a \in A$. Show that this makes $\text{Hom}_\Lambda(A, X)$ into a left Γ-module.

2. (Lemma 47 in the videos) Let Λ be an arbitrary ring and let $A_1, A_2, B_1, B_2 \in \text{Mod}\Lambda$.

   (a) Show that there is an isomorphism of abelian groups

   $$\alpha : \text{Hom}_\Lambda(A_1, B_1) \oplus \text{Hom}_\Lambda(A_1, B_2) \to \text{Hom}_\Lambda(A_1, B_1 \oplus B_2)$$

   given by $\alpha(f, g)(a) := (f(a), g(a))$ for $a \in A$.

   (b) Show that there is an isomorphism of abelian groups

   $$\beta : \text{Hom}_\Lambda(A_1, B_1) \oplus \text{Hom}_\Lambda(A_2, B_1) \to \text{Hom}_\Lambda(A_1 \oplus A_2, B_1)$$

   given by $\beta(f, g)(a_1, a_2) := f(a_1) + g(a_2)$ for $a_1 \in A_1$ and $a_2 \in A_2$.

   (c) Let $\Gamma = \text{End}_\Lambda(A_1)^{op}$. Show that the isomorphism α in part (a) is an isomorphism of Γ-modules.

3. (From the proof of Proposition 48 in the videos) Let Λ be a ring and let $A, B, C, A', B', C' \in \text{Mod}\Lambda$.

   (a) Suppose there is a commutative diagram

   $$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{t_1} & B & \xrightarrow{t_2} & C \\ & & & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A' & \xrightarrow{s_1} & B' & \xrightarrow{s_2} & C' \end{array}$$

   with both the top and bottom rows exact. Show that there exists a unique $f : A \to A'$ so that $g \circ t_1 = s_1 \circ f$. Moreover, show that if g and h are both isomorphisms, then so is f.

(b) Suppose there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{t_1} & B & \xrightarrow{t_2} & C & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ A' & \xrightarrow{s_1} & B' & \xrightarrow{s_2} & C' & \longrightarrow & 0 \end{array}$$

with both the top and bottom rows exact. Show that there exists a unique $h : C \to C'$ so that $h \circ t_2 = s_2 \circ g$. Moreover, show that if f and g are both isomorphisms, then so is h.

4. (From the proof of Lemma 49 in the videos) Let $F : \mathscr{C} \to \mathscr{D}$ be an equivalence of categories. Let X and Y be objects in $\mathscr{C}$. Show that $X \cong Y$ if and only if $F(X) \cong F(Y)$.