MA3203 - Problem Set 13 Answers and Hints

(1b) Define $\varphi : \oplus_{i=1}^{m} \Lambda e_i \to \Lambda e$ and $\psi : \Lambda e \to \oplus_{i=1}^{m}$ by

$$\varphi(x_1 e_1, \ldots, x_m e_m) = \sum_{i=1}^{m} x_i e_i e, \qquad \psi(xe) = (xe_1, \ldots, xe_m).$$

It is straightforward to show that φ and ψ are homomorphisms of Λ modules. We then have

$$\psi \circ \varphi(x_1 e_1, \ldots, x_m e_m) = \left(\sum_{i=1}^{m} x_1 e_1 e, \ldots, \sum_{i=1}^{m} x_m e_m e \right) = (x_1 e_1, \ldots, x_m e_m)$$

since the e_i are orthogonal idempotents. Likewise,

$$\varphi \circ \psi(xe) = \sum_{i=1}^{m} xe_i e = xe$$

for the same reason.

(1c) It is immediate from (1b) that if e is not primitive, then Λe is not indecomposable. Thus suppose e is primitive and write $\Lambda e \cong M \oplus N$. Let $q_M : \Lambda e \twoheadrightarrow M$ be the projection map and $\iota_M : M \hookrightarrow \Lambda e$ be the inclusion map, and likewise for N. Then $\iota_M \circ q_M + \iota_N \circ q_N = \mathrm{Id}_M$. In particular, $e = \iota_M \circ q_M(e) + \iota_N \circ q_N(e)$ and both $\iota_M \circ q_M(e)$ and $\iota_N \circ q_N(e)$ are idempotents. Thus, since e is primitive, we have without loss of generality that $\iota_N \circ q_N(e) = 0$ and $\iota_M \circ q_M(e) = e$. This means $N = 0$ and $M \cong \Lambda e$, so Λe is indecomposable.

(2) Since e is primitive, we have that Λe is indecomposable and so $\Lambda e / \mathrm{rad} \Lambda e$ is simple. Now if $eS \neq 0$, then there is a nonzero morphism $\Lambda e \to S$ which induces an isomorphism $\Lambda e / \mathrm{rad} \Lambda e \to S$. This map is then a projective cover. Otherwise, there is no nonzero morphism $\Lambda e \to S$.