MA3203 - Problem Set 13 (Idempotents)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise stated.

1. (Proposition 39bc from the videos) Let Λ be a left artinian ring.

   (b) Let $e = e_1 + \cdots + e_m \in \Lambda$ be a sum of orthogonal idempotents. Show that $\Lambda e \cong \bigoplus_{i=1}^m \Lambda e_i$.

   (c) Let $e \in \Lambda$ be a nonzero idempotent. Show that Λe is indecomposable if and only if e is primitive.

2. [1, Exercise 4.7] Let Λ be a left artinian ring and let $e \in \Lambda$ be a nonzero primitive idempotent. Let S be a simple (left) Λ-module. Show that there is a projective cover $\Lambda e \to S$ if and only if $eS \neq 0$.

3. Let Λ be a finite-dimensional K-algebra and let $e \in \Lambda$ be a nonzero idempotent.

   (a) Show that $e\Lambda e$ is a (finite-dimensional) K-algebra.

   (b) Show that $e = 1_{e\Lambda e}$.

   (c) Show that e is primitive if and only if $e\Lambda e$ is local. *Hint: recall that $e\Lambda e$ is local if and only if its only idempotents are 0 and $1_{e\Lambda e}$.*

   (d) e is called a *central idempotent* if $e\rho = \rho e$ for all $\rho \in \Lambda$. Show that if e is a central element, then $1 - e$ is also a central idempotent.

   (e) Show that if e is a central idempotent, then $e\Lambda$ and $(1-e)\Lambda$ are **left** Λ-modules.

   (f) Show that if e is a central idempotent, then $e\Lambda$ and $(1-e)\Lambda$ are (finite-dimensional) K-algebras and $\Lambda \cong e\Lambda \oplus (1-e)\Lambda$ (as K-algebras).

   (g) We say Λ is *connected* if and only if there do not exist nonzero K-algebras Λ' and Λ'' so that $\Lambda \cong \Lambda' \oplus \Lambda''$ (as K-algebras). Show that Λ is connected if and only if 0 and 1_Λ are its only central idempotents. *Hint: If $\Lambda = \Lambda' \oplus \Lambda''$, then $1_\Lambda = (1_{\Lambda'}, 1_{\Lambda''})$.*

4. *Note: This problem relies on Problem 3.* Let Q be a quiver. We will show that KQ is connected (as a K-algebra) if and only if Q is connected (as a graph).

   (a) Suppose Q is not connected and let Q' be a connected component of Q. Show that $\sum_{i \in Q'_0} e_i$ is a nonzero central idempotent in KQ and conclude that KQ is not connected.

   (b) Now suppose KQ is not connected and let $c \in KQ$ be a nonzero central idempotent. Show that for all $i \in Q_0$, the element $e_i c e_i \in KQ$ is idempotent.

   (c) Conclude that either $e_i c e_i = e_i$ or $e_i c e_i = 0$. *Hint: Use (3c).*

   (d) Let $Q'_0 = \{i \in Q_0 : e_i c e_i = e_i\}$ and $Q''_0 = \{j \in Q_0 : e_j c e_j = 0\}$. For any $i \in Q'_0$ and $j \in Q''_0$, show that $e_i KQ e_j = 0 = e_j KQ e_i$ and conclude that Q contains neither an arrow $i \to j$ nor an arrow $j \to i$.

   (e) Show that both Q'_0 are Q''_0 nonempty and conclude that Q is not connected.

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.