MA3203 - Problem Set 14 Hints and Answers

(3a) Let $x, y, z \in P$ and suppose $\mathrm{Hom}_{\mathscr{P}}(x, y) \neq \emptyset \neq \mathrm{Hom}_{\mathscr{P}}(y, z)$. This means $x \leq y$ and $y \leq z$. Therefore $x \leq z$ since partial orders are transitive. Thus for $* : x \to y$ and $* : y \to z$, we have $* \circ * = * : x \to z$. It is straightforward to show that this composition law is associative.

Now given $x \in P$, we have $x \leq x$ since partial orders are reflexive. The identity morphism of x is then the unique morphism $* : x \to *$. This shows that $\mathscr{P}$ is well-defined.

Finally, let $x \neq y \in P$. Then at least one of $\mathrm{Hom}_{\mathscr{P}}(x, y)$ and $\mathrm{Hom}_{\mathscr{P}}(y, x)$ is empty since partial orders are anti-symmetric. It follows that the only isomorphisms in $\mathscr{P}$ are the identity morphisms.

(3b) The composition law in $\mathscr{G}$ is associative because group multiplication is associative. This means given $f, g, h \in \mathrm{Hom}_{\mathscr{G}}(*, *)$, we have $(f \circ g) \circ h = (fg)h = f(gh) = f \circ (g \circ h)$. Moreover, let 1 be the identity of G. Then for $f : * \to *$ in $\mathscr{G}$ we have $1 \circ f = 1f = f = f1 = f \circ 1$, and so 1 is the identity of $*$ in $\mathscr{G}$. This shows that $\mathscr{G}$ is well-defined.

Finally, for all $f \in \mathrm{Hom}_{\mathscr{G}}(*, *)$, let f^{-1} be the inverse of f in G. Then $f \circ f^{-1} = ff^{-1} = 1 = f^{-1}f = f^{-1} \circ f$. This shows that every morphism in $\mathscr{G}$ is an isomorphism.

(5a) Let A, B, C be objects of $\mathscr{C}$. Suppose there are morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and $g_1, g_2 \in \mathrm{Hom}_{\mathscr{C}}(C, A)$ so that f is an isomorphism and $f \circ g_1 = f \circ g_2$. Let f^{-1} be the inverse of f. Then since $\circ$ is associative, we have

$$g_1 = \mathrm{Id}_B \circ g_1 = f^{-1} \circ f \circ g_1 = f^{-1} \circ f \circ g_2 = \mathrm{Id}_B \circ g_2 = g_2.$$

We conclude that f is a monomorphism. The argument that f is an epimorphism is analogous.

(5b) Let A, B be objects of $\mathscr{C}$ and let $f : A \to B$ be surjective. Let C be another object of $\mathscr{C}$ and consider $g_1, g_2 \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ so that $g_1 \circ f = g_2 \circ f$. Now let $x \in B$ (we can talk about elements because we are working with sets!). Since f is surjective, there exists $y \in A$ so that $f(y) = x$. We then have

$$g_1(x) = g_1 \circ f(y) = g_2 \circ f(y) = g_2(x).$$

We conclude that $g_1 = g_2$ and so f is an epimorphism. The argument that if f is injective then it is a monomorphism is similar.

(5c) (i) Let $f : A \to B$ be a monomorphism of abelian groups. Let $\iota : \ker(f) \to A$ and $0 : \ker(f) \to A$ be the inclusion map and the trivial map, respectively. Then $f \circ \iota = f \circ 0$. Since f is a monomorphism, we conclude that $\iota = 0$; i.e., that $\ker(f) = 0$.

(ii) Let $f : A \to B$ be an epimorphism of abelian groups. Let $q : B \to B/\mathrm{Im}(f)$ and $0 : B \to B/\mathrm{Im}(f)$ be the quotient map and the trivial map, respectively. Then $q \circ f = 0 \circ f$. Since f is an epimorphism, we conclude that $q = 0$; i.e., that $\mathrm{Im}(f) = B$.

(iii) We know that f is an isomorphism of abelian groups if and only if it is a bijection. By (i) and (ii), this is equivalent to f being both a monomorphism and an epimorphism.

(5d) Denote the inclusion map $\iota : \mathbb{Z} \to \mathbb{Q}$. Let R be a ring and let $g_1, g_2 : \mathbb{Q} \to R$ so that $g_1 \circ \iota = g_2 \circ \iota$. Let $z \in \mathbb{Q}$ and write $z = x/y$ with $x, y \in \mathbb{Z}$. Then

$$g_1(z) = g_1(x)(g_1(y))^{-1} = g_1 \circ \iota(x)(g_1 \circ \iota(y))^{-1}$$

$$= g_2 \circ \iota(x)(g_2 \circ \iota(y))^{-1} = g_2(x)g_2(y))^{-1} = g_2(z).$$

We conclude that $g_1 = g_2$ and so ι is an epimorphism. We know ι is a monomorphism as well by (b).

(5e) Let $f : R \to S$ be a monomorphism of rings and let $x \in \ker(f)$. Now define $g, g_x : \mathbb{Z}[t] \to R$ by $g(t) := 0$ and $g_x(t) := x$. Then $f \circ g = f \circ g_x$ is the morphism $\mathbb{Z}[t] \to S$ which sends 1 to 1 and sends t to 0. Therefore, since f is a monomorphism, we have $g = g_x$ and so $x = 0$. We conclude that $\ker(f) = 0$ and f is injective.

(7b) We need to show that $\mathrm{Hom}_{\mathscr{C}}(-, A)$ preserves the composition law. Thus let B, C, D be objects of $\mathscr{C}$ and let $f \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(C, D)$. Then for $h \in \mathrm{Hom}_{\mathscr{C}}(D, A)$, we have

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = (g^*(h)) \circ f = f^* \circ g^*(h).$$

(8) Consider the sequence

$$0 \to \mathrm{Hom}_\Lambda(M, A) \xrightarrow{f_*} \mathrm{Hom}_\Lambda(M, B) \xrightarrow{g_*} \mathrm{Hom}_\Lambda(M, C).$$

First let $h_1, h_2 \in \mathrm{Hom}_\Lambda(M, A)$ and suppose $f_*(h_1) = f_*(h_2)$; that is, that $f \circ h_1 = f \circ h_2$. Since f is injective, (5b) then implies that $h_1 = h_2$, and so f_* is a monomorphism. (5c) then implies that f_* is injective.

Now, the fact that $f \circ g = 0$ implies that $(f \circ g)_* = 0$, and so $\mathrm{Im}(f_*) \subseteq \ker(g_*)$. Thus let $h \in \ker(g_*)$. This means for all $x \in M$, we have $g \circ h(x) \in M$, and so $h(x) \in \ker(g) = \mathrm{Im}(f)$. Since f is injective, there then exists a unique $\tilde{x} \in A$ so that $h(x) = f(\tilde{x})$.[1] We can then define $\tilde{h} : M \to A$ by $\tilde{h}(x) = \tilde{x}$. One can then show that $\tilde{h}$ is a morphism of Λ-modules and that $h = f_*(\tilde{h})$. This means $\ker(g_*) \subseteq \mathrm{Im}(f_*)$ and so $\ker(g_*) = \mathrm{Im}(g_*)$.

[1]Note: sometimes, it is assumed that $A \subseteq B$ and f is an inclusion map, in which case this step is not necessary.

We have thus far shown that $\mathrm{Hom}_\Lambda(M, -)$ is left exact. Moreover, we recall that every surjective morphism of Λ-modules $g : B \twoheadrightarrow C$ is part of the short exact sequence

$$0 \to \ker g \to B \xrightarrow{g} C \to 0.$$

Therefore, $\mathrm{Hom}_\Lambda(M, -)$ is exact if and only if $g_* : \mathrm{Hom}_\Lambda(M, B) \to \mathrm{Hom}_\Lambda(M, C)$ is surjective whenever g is surjective. That is; given a surjection $g : B \twoheadrightarrow C$ and a morphism $f : M \to C$, there exists a morphism $h : M \to B$ so that $f = g \circ h$. This is the definition of what it means for M to be projective.

(9a) Let $f, g : V \to V'$ be two morphisms of vector spaces. Then $f = g$ if and only if $f(x) = g(x)$ for all $x \in V$; i.e., if and only if $U(f) = U(g)$. This shows that U is faithful.

To see that U is not full, consider the morphism of sets $f : K \to K$ given by $f(x) = 1_K$ for all x. Then f is not a morphism of vector spaces.

To see that U is not dense, consider first the case where $|K| \neq 2$. Then there is no K-vector space V with $|V| = 2$. Therefore, if S is a set with 2 elements, there is no vector space V so that $U(V) \cong S$. Otherwise, $|K| = 2$ and we instead consider a set with 3 elements.

(9b) Every vector space has a basis, so given $V \in Vec(K)$ there exists $S \in Set$ so that $B(S) = V$. (Note in particular that if V is 0-dimensional, then $V = B(\emptyset)$.) This shows that B is dense. Moreover, let S, S' be sets and $f, f' : S \to S'$. Then $B(f) = B(f')$ if and only if $f(x) = f'(x)$ for all $x \in S$. This means that B is faithful. (Again, it is possible that $S = \emptyset$, in which case f is the "empty function" and $B(f)$ is the unique morphism sending the 0-dimensional vector space $B(S)$ to $B(S')$.)

To see that B is not full, let $S \neq \emptyset$. Then $\mathrm{Hom}_{Set}(S, \emptyset) = \emptyset$, but there is a morphism of vector spaces $B(S) \to B(\emptyset)$.

(9c) This statement just says (in a confusing way) that given two vector spaces V' and V and a fixed basis S of V', every linear map $V \to V'$ is completely and uniquely determined by where it sends S.

(10a) We know that finite dimensional vector spaces are isomorphic if and only if they have the same dimension. Therefore, given $A, B \in \mathscr{K}$, we have $A \cong B$ if and only if $A = B$. Moreover, we know that every finite dimensional vector space is isomorphic to K^n for some n, meaning that the inclusion functor ι is dense. It is clear that ι is full and faithful since $\mathscr{K}$ is a full subcategory. We conclude that $\mathscr{K}$ is a skeleton of $vec(K)$.

(10b) The elements of $vec(K)$ do not form a set. For example, let S be any set and let K_S be the 1-dimensional vector space with basis the formal symbol S. This means we are not considering the elements of S as a basis, but rather elements of K_S are of the form λS for $\lambda \in K$. Now given two sets $S \neq T$, we have $K_S \neq K_T$ (even though $K_S \cong K_T$). As the collection of all sets is not a set, this shows that the objects of $vec(K)$ do not form a set. On the other hand, the objects of $\mathscr{K}$ are in bijection with the natural numbers, meaning they do form a set.