# MA3203 - Problem Set 14 (Categories and Functors)

This is the last new problem set before the break, so it is intended to be completed over several days. The problems in Parts I and II only require knowledge from the videos for Friday 19.3, while some of the problems in Part III require knowledge from the videos for Tuesday 23.3 as well.

**I. Examples of categories**

1. (a) Show that there is a well-defined category *Set* whose objects are sets, morphisms are the usual maps between sets, and composition is the usual composition.

   (b) Show that there is a well-defined category *Ring* whose objects are rings (with 1), morphisms are ring homomorphisms (sending 1 to 1), and composition is the usual composition.

   (c) Recall that a poset $P$ is a set $S$ together with a reflexive, antisymmetric, transitive relation $\leq$ (meaning that (1) for all $x, y, z \in P$, we have $x \leq x$ (2) if $x \leq y$ and $y \leq x$, then $x = y$, and (3) if $x \leq y$ and $y \leq z$ then $x \leq z$). An order-preserving map of posets $P \to Q$ is a map $f : P \to Q$ of sets so that if $x \leq y$ in $P$ then $f(x) \leq f(y)$ in $Q$. Show that there is a well-defined category *Pos* whose objects are posets, morphisms are order-preserving maps, and composition is the usual composition.

2. Let $\mathscr{C}$ be a category and let $A$ be an object of $\mathscr{C}$. Show that there is a well-defined category $\mathscr{C}/A$ whose objects are pairs $(B, f)$ with $B$ an object of $\mathscr{C}$ and $f \in \operatorname{Hom}_{\mathscr{C}}(B, A)$, whose morphisms are given by $\operatorname{Hom}_{\mathscr{C}/A}((B, f), (B', f')) = \{g \in \operatorname{Hom}_{\mathscr{C}}(B, B') : f' \circ g = f\}$, and whose composition law is given by composition in $\mathscr{C}$.[1]

3. (a) Let $P$ be a poset and let $\{*\}$ be a set with one element. Show that there is a well-defined category $\mathscr{P}$ whose objects are the elements of $P$ and whose morphisms are given by

   $$\operatorname{Hom}_{\mathscr{P}}(x, y) = \begin{cases} \{*\} & x \leq y \\ \emptyset & x \not\leq y. \end{cases}$$

   What is the composition law in this category? What are the identity morphisms? Which morphisms are isomorphisms?

   (b) Let $G$ be a group and let $\{*\}$ be a set with one element. Define a category $\mathscr{G}$ with a single object $*$, morphisms $\operatorname{Hom}_{\mathscr{G}}(*, *) = G$, and composition given by $g \circ h := gh$ for all $g, h \in \operatorname{Hom}_{\mathscr{G}}(*, *)$. Show that $\mathscr{G}$ is a well-defined category and that every $g \in \operatorname{Hom}_{\mathscr{G}}(*, *)$ is an isomorphism.

[1] This is sometimes called the *over category* of $\mathscr{C}$ over $A$.

## II. Properties of objects and morphisms

4. Show that $\mathbb{Z} \cong \mathbb{Q}$ in the category *Set*, but (with the usual notion of $\leq$) $\mathbb{Z} \not\cong \mathbb{Q}$ in the category *Pos*.

5. Let $\mathscr{C}$ be a category and let $A, B$ be objects of $\mathscr{C}$. A morphism $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$ is called a *monomorphism* if for any object $C \in \mathscr{C}$ and any pair of morphisms $g_1, g_2 \in \operatorname{Hom}_{\mathscr{C}}(C, A)$ we have $f \circ g_1 = f \circ g_2$ if and only if $g_1 = g_2$. Likewise, $f$ is called an *epimorphism* if for any object $C \in \mathscr{C}$ and any pair of morphisms $g_1, g_2 \in \operatorname{Hom}_{\mathscr{C}}(B, C)$ we have $g_1 \circ f = g_2 \circ f$ if and only if $g_1 = g_2$.

   (a) Let $\mathscr{C}$ be arbitrary. Show that any isomorphism in $\mathscr{C}$ is both a monomorphism and an epimorphism.

   (b) Suppose that (a) the objects of $\mathscr{C}$ are also sets (possibly with some additional structure), (b) if $A$ and $B$ are objects of $\mathscr{C}$ and $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$, then $f$ is also a map of sets from $A$ to $B$, and (c) composition in $\mathscr{C}$ is the usual rule for composition (these properties are true for example in the categories *Set*, *Ring*, *Ab*, $Mod\Lambda$ for any ring $\Lambda$, etc.)[2]. Let $f$ be a morphism in $\mathscr{C}$. Show that if $f$ is injective, then it is a monomorphism and that if $f$ is surjective, then it is an epimorphism.

   (c) Let $f$ be a morphism in *Ab*. Show that

      i. $f$ is a monomorphism if and only if it is injective.

      ii. $f$ is an epimorphism if and only if it is surjective.

      iii. $f$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.

   (d) Show that the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in the category *Ring*.

   (e) (Challenge) Show that every monomorphism in *Ring* is injective (but note that we have already shown that not every epimorphism in *Ring* is surjective!).

6. Let $\mathscr{C}$ be a category. An object $I$ of $\mathscr{C}$ is called *initial* if for any object $A \in \mathscr{C}$ the set $\operatorname{Hom}_{\mathscr{C}}(I, A)$ contains a single element. Likewise, an object $T$ of $\mathscr{C}$ is called *terminal* if for any object $A \in \mathscr{C}$ the set $\operatorname{Hom}_{\mathscr{C}}(A, T)$ contains a single element. If the same object is both initial and terminal, it is called a *zero object*.

   (a) Show that the trivial group is a zero object in the category *Ab* (and more generally in $Mod\Lambda$ for any ring $\Lambda$).

   (b) Show that the 0 ring is terminal in *Ring* and $\mathbb{Z}$ is initial in *Ring*.

   (c) (Challenge) Show that if $A$ and $B$ are both initial (resp. both terminal) in some category $\mathscr{C}$, then $A$ and $B$ are isomorphic. *Hint: consider the composition* $A \to B \to A$.

[2] Categories with this property are called *concrete categories*.

**III. Functors and morphisms of functors**

7. Let $\mathscr{C}$ be any category and let $A$ be an object in $\mathscr{C}$.

   (a) Show that there is a covariant functor $\mathrm{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} \to Set$ which sends an object $B$ of $\mathscr{C}$ to the set $\mathrm{Hom}_{\mathscr{C}}(A,B)$ and which sends a morphism $f : B \to C$ in $\mathscr{C}$ to the map $f_* : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$ given by $f_*(g) := f \circ g$.

   (b) Show that there is a contravariant functor $\mathrm{Hom}_{\mathscr{C}}(-,A) : \mathscr{C} \to Set$ which sends an object $B$ of $\mathscr{C}$ to the set $\mathrm{Hom}_{\mathscr{C}}(B,A)$ and which sends a morphism $f : B \to C$ in $\mathscr{C}$ to the map $f^* : \mathrm{Hom}_{\mathscr{C}}(C,A) \to \mathrm{Hom}_{\mathscr{C}}(B,A)$ given by $f^*(g) := g \circ f$.

8. Let $\Lambda$ be a ring and let $Mod\Lambda$ be the category of left $\Lambda$-modules. A covariant functor $F : Mod\Lambda \to Ab$ is called *left exact* if for any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   of $\Lambda$-modules the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

   is exact (this means that $F(f)$ is injective and $\ker F(g) = \mathrm{Im} F(f)$). If in addition $F(g)$ is surjective, then $F$ is called *exact*.

   Let $M$ be a left $\Lambda$-module. Show that $\mathrm{Hom}_\Lambda(M,-) : Mod\Lambda \to Ab$ is always left exact and is exact if and only if $M$ is projective.

9. Let $K$ be a field. Define a functor $U : Vec(K) \to Set$ by $U(V) := V$ and $U(f) := f$ for all objects $V$ and morphisms $f$ in $Vec(K)$.[3] Define a second functor $B : Set \to Vec(K)$ which sends a set $S$ to the $K$-vector space with basis $S$ and sends a morphism of sets $f : S \to S'$ to the unique linear map $\tilde{f}$ which is equal to $f$ on the basis $S$.

   (a) Show that $U$ is faithful, but neither full nor dense.

   (b) Show that $B$ is dense and faithful, but not full.

   (c) For any $K$-vector space $V$ and set $S$, show that there is an isomorphism of sets (that is, a bijection)

$$\mathrm{Hom}_{Vec(K)}(B(S), V) \cong \mathrm{Hom}_{Set}(S, U(V))).$$

   (d) Show that, for any set $S$, there is an isomorphism of functors

$$\eta : \mathrm{Hom}_{Vec(K)}(B(S), -) \cong \mathrm{Hom}_{Set}(S, -)$$

   so that for any $K$-vector space $V$, the morphism $\eta_V$ is the bijection you found in part (c).

[3] This is usually called a *forgetful functor*.

(e) (Challenge) Show that, for any $K$-vector space $V$, there is an isomorphism of **contravariant** functors

$$\eta' : \mathrm{Hom}_{Vec(K)}(B(-), V) \cong \mathrm{Hom}_{Set}(-, U(V))$$

so that for any set $S$, the morphism $\eta'_S$ is the the bijection you found in part (c).

(f) (Nothing to prove for this part) If you are interested in why these properties are important, check out the Wikipedia page on "adjoint functors".

10. Let $\mathscr{C}$ be a category. A subcategory $\mathscr{S} \subseteq \mathscr{C}$ is called a *skeleton* of $\mathscr{C}$ if the inclusion functor $\iota : \mathscr{S} \to \mathscr{C}$ is an equivalence of categories and for any two objects $A, B \in \mathscr{S}$, there is an isomorphism $A \cong B$ if and only if $A = B$.

    (a) Let $vec(K)$ be the category of finite-dimensional vector spaces over a field $K$. Let $\mathscr{K}$ be the full subcategory of $vec(K)$ whose objects are those vector spaces of the form $K^n$ for some nonnegative integer $n$. Show that $\mathscr{K}$ is a skeleton of $vec(K)$.

    (b) Do the objects of $vec(K)$ form a set? What about the objects of $\mathscr{K}$?