Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3203
Ring theory
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Exercise set 6

1 Given the following quivers Q and ideals I of kQ, compute the radicals and tops of Λe_i for all $i \in Q_0$, where $\Lambda = kQ/I$.

a) Q: $\begin{array}{ccccc} & & & & 3 \\ & & & \swarrow & \\ 1 & \longrightarrow & 2 & & \\ & & & \nwarrow & \\ & & & & 4 \end{array}$, $I = (0)$;

b) Q: $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$, $I = \operatorname{rad}^2 kQ$;

c) Q: $1 \xrightarrow{\alpha} 2 \overset{\gamma}{\underset{\beta}{\rightrightarrows}} 3$, $I = \langle \beta\alpha - \gamma\alpha \rangle$;

d) Q: $\begin{array}{ccc} 2 & \xrightarrow{\beta} & 3 \\ \alpha \uparrow & & \downarrow \gamma \\ 1 & \xleftarrow[\delta]{} & 4 \end{array}$, $I = \langle \gamma\beta, \delta\gamma, \alpha\delta \rangle$.

2 Given the following quivers Q and ideals I of kQ, compute the radicals and tops of the given kQ/I-modules M.

a) Q: $\begin{array}{ccc} & & 3 \\ & \nearrow & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$, $I = (0)$, M: $\begin{array}{ccc} & & k \\ & \overset{1}{\nearrow} & \uparrow \begin{pmatrix} 0 & 1 \end{pmatrix} \\ k & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & k^2 \end{array}$;

b) Q: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \circlearrowleft \gamma$, $I = \langle \gamma^3 \rangle$, M: $k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}} k^3 \circlearrowleft \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

3 Let $f : M \to N$ be a module homomorphism. We call $s : N \to M$ a *section* of f if $f \circ s = 1_N$. We call $r : N \to M$ a *retraction* of f if $r \circ f = 1_M$.

a) Prove that f admits a retraction r if and only if f is injective and $N = \text{Im } f \oplus L$, where L is some submodule of N.

b) Prove that f admits a section s if and only if f is surjective and $M = L \oplus \ker f$, where L is some submodule of M.

c) Given a short exact sequence of modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

show that f admits a retraction if and only if g admits a section.

4 **a)** (Baer's criterion) Let Λ be a ring. Show that a module I is injective if and only if for every ideal $\mathfrak{a}$ of Λ, any Λ-module homomorphism $f : \mathfrak{a} \to I$ can be extended to a Λ-module homomorphism $f' : \Lambda \to I$, that is, such that $f'|_{\mathfrak{a}} = f$.

(Show hint)

b) Conclude that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.