

# Notes on basic algebras

Definition:  $\Lambda$  artinian. We say  $\Lambda$  is basic if  
 $\Lambda = P_1 \oplus \dots \oplus P_n$  with  $P_i$  indec. (proj.)  
 $\Rightarrow P_i \not\cong P_j$  for  $i \neq j$ .

Remark:  $\Lambda/\mathfrak{r} = S_1 \oplus \dots \oplus S_n$  semisimple  
 $P(S_i) \rightarrow S_i$  proj. cover  
 $\Rightarrow \Lambda = P(S_1) \oplus \dots \oplus P(S_n)$   
 $P(S_i) \cong P(S_j) \Leftrightarrow S_i \cong S_j$   
 $\Rightarrow \Lambda$  basic  $\Leftrightarrow S_i \not\cong S_j$   $i \neq j$ .

Definition: A set  $\{e_1, \dots, e_n\}$  of idempotents is complete if  $e_1 + \dots + e_n = 1$   
 It is orthogonal if  $e_i e_j = \begin{cases} e_i & i=j \\ 0 & \text{else} \end{cases}$

An idempotent  $e$  is called primitive if  $e \neq e_1 e_2$   
 (i.e.  $\Lambda e$  indecomposable)  $e_1, e_2$  orthogonal  $\neq 0$

Remark:  $\Lambda = P_1 \oplus \dots \oplus P_n$  indec. proj  
 gives rise to a complete set of primitive  
 orthogonal idempotents (spoi)  $e_i: \Lambda \rightarrow P_i$   
 Conversely  $\{e_1, \dots, e_n\}$  (spoi) gives a  
 decomposition  $\Lambda = \Lambda e_1 \oplus \dots \oplus \Lambda e_n$ .

$\Rightarrow \Lambda$  basic  $\Leftrightarrow \Lambda e_i \not\cong \Lambda e_j$  for all  $e_i \neq e_j \in \{e_1, \dots, e_n\}$  (spoi).

Examples ①  $\Lambda = \begin{pmatrix} k & k \\ k & k \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$

However,  $\begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \not\cong \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix} \Rightarrow \Lambda$  not basic

Simple  $\Lambda$ -modules  
 $\Rightarrow$  indecomposable

② Any quiver algebra  $\Lambda = k \text{ @ } \Gamma$  is basic  
 $\Gamma$  admissible

Have  $\Lambda/\mathfrak{r} \cong S_1 \oplus \dots \oplus S_n$  &  $S_i \not\cong S_j$  if  $i \neq j$ .

Lemma:  $D$  division  $n \cdot k$  algebra,  $k$ -alg. closed,  
 $\Rightarrow D \cong k$

Proof: By contradiction, let  $a \in D \setminus k$ .

Let  $d = \dim_k D$

$\rightarrow \{1, a, \dots, a^d\}$  linearly dependent

$\Rightarrow a$  is a root of a polynomial  $f(x)$ .

Choose  $f$  with minimal degree

$k$  alg. closed  $\Rightarrow f$  has a root  $\lambda \in k$

$\Rightarrow f(x) = g(x)(x - \lambda)$

$f$  minimal  $\Rightarrow g(a) \neq 0$

$\Rightarrow a = \lambda \in k \rightarrow \leftarrow$

From now on:  $K$  alg. closed

(2)

Proposition:  $A$  f.d.  $K$ -algebra.  $K$  alg. closed

a)  $A$  basic  $\Leftrightarrow A/\text{rad} A \cong K \times \dots \times K$

b)  $A$  basic  $\Rightarrow$  every simple module is 1-dim'l.

Proof a) Decompose  $1 = \sum_{i=1}^n e_i$ ;  $\{e_1, \dots, e_n\}$  c.s.p.o.i.

$\Rightarrow \bar{e}_j = e_j + \text{rad} A$  is an idempotent in  $A/\text{rad} A = \text{top} A$   
&  $\text{top} A \bar{e}_j$  is a simple  $\text{top} A$ -module.

$\Rightarrow \text{top} A \cong \text{top} A \bar{e}_1 \oplus \dots \oplus \text{top} A \bar{e}_n$   
 $\hookrightarrow$  simples

Have  $\text{top} A \bar{e}_j \cong \text{top} A \bar{e}_i \Leftrightarrow \text{top} A \bar{e}_i \cong \text{top} A \bar{e}_j$

$\Rightarrow$   $A$  basic  $\Rightarrow$   $\text{top} A$  basic

Schur's Lemma:  $\text{Hom}_{\text{top} A}(\text{top} A \bar{e}_i, \text{top} A \bar{e}_j) = 0$   $i \neq j$   
&  $\text{End}_{\text{top} A}(\text{top} A \bar{e}_j) = \text{division algebra} \cong K$ .  $\forall j$

Thus, given  $b \in \text{top} A$   $j \leq n \Rightarrow b_j: \text{top} A \bar{e}_j \rightarrow \text{top} A$

induces  $b_j: \text{top} A \bar{e}_j \rightarrow \text{top} A \bar{e}_j$   
and

$K$ -alg. hom.

$\sigma_j: \text{top} A \rightarrow \text{End}_{\text{top} A} \text{top} A \bar{e}_j \cong K$   
 $b \mapsto b_j$

$\Rightarrow$  Get a  $K$ -algebra homom.  $\sigma: \text{top} A \rightarrow \dots$

$\sigma: \text{top} A \rightarrow \text{End}_{\text{top} A} \text{top} A \bar{e}_1 \times \dots \times \text{End}_{\text{top} A} \text{top} A \bar{e}_n$   
 $\cong K \times \dots \times K$

$b \mapsto (\sigma_1(b), \dots, \sigma_n(b))$

$\sigma$  injective  $\Rightarrow \sigma$  bijective by comparing dimensions.

$\Leftrightarrow \text{top } 1 \cong k \times \dots \times k$  commutative  
 $\Rightarrow \text{top } 1 e_i \not\cong \text{top } 1 e_j \quad i \neq j$

b)  $\Lambda$  basic  $\Rightarrow \text{top } 1 \cong k \times \dots \times k \cong S_1 \oplus \dots \oplus S_n$   
 $\Rightarrow S_i \cong k \quad \forall i \Rightarrow \dim_k S_i = 1 \quad \forall i \quad \square$

Definition:  $\Lambda$  f.d.  $k$ -alg.  $\{e_1, \dots, e_n\}$  cspoi  
 A basic algebra  $\Lambda^b$  associated to  $\Lambda$  is defined as

$\Lambda^b = e_1 \Lambda e_n$  where  
 $e_1 = e_{j_1} + \dots + e_{j_s}$  and  $e_{j_1}, \dots, e_{j_s}$  are  
 chosen s.t.  $\Lambda e_{j_s} \not\cong \Lambda e_{j_t} \quad s \neq t$   
 and each  $\Lambda e_{j_s}$  is iso to one module  
 $\Lambda e_{j_1}, \dots, \Lambda e_{j_s}$ .

Idea:  $\Lambda^b$  picks only one copy of repeating  
 indecomposable projectives in the  
 decomposition of  $\Lambda$ .

Example:  $\Lambda = M_n(k) \quad e_{ii} = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 1 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$  position  $(i, i)$

$\Rightarrow \{e_{11}, \dots, e_{nn}\}$  cspoi but  $\Lambda e_{ii} \cong \Lambda e_{jj} \quad \forall i, j$

$\Rightarrow e_1 = e_1 \quad \& \quad \Lambda^b = e_1 \Lambda e_1 \cong k$

Remark:  $\Lambda$  basic  $\Rightarrow \Lambda^b = \Lambda$ .

Lemma: a) The idempotent  $e_1 \in A^b$  is the identity in  $A^b$  &  $A^b \cong \text{End}_k(Ae_1 \oplus \dots \oplus Ae_j a)$   
k-alg. iso

b)  $A^b$  does not depend on the choice of idempotents up to iso.

Proof: a) Exo set 3.  $Ae_1 \cong Ae_j \oplus \dots \oplus Ae_j a$   
 &  $\text{End}_k(Ae_1) \cong e_1 A e_1$

b) By KRS theorem, any decomposition of  $1$  &  $A^b$  is unique up to iso.

Lemma:  $A^b$  is a basic algebra.

Proof: Let  $\{e_1, \dots, e_n\}$  be a c.s.p.i. of  $1$   
 $\Rightarrow \{e_{j_1}, \dots, e_{j_a}\}$  are orthogonal idemp. in  $A^b$   
 &  $A^b = A^b e_1 = A^b e_{j_1} \oplus \dots \oplus A^b e_{j_a}$   
 $\Rightarrow$  the set is complete.

To show they are primitive, need to show  $A^b e_{j_i}$  indecomposable.

Have  $\text{End}(A^b e_{j_i}) \cong e_{j_i} A^b e_{j_i}$  is local.

Finally, clearly  $A^b e_{j_s} \not\cong A^b e_{j_t}$   $s \neq t$   
 $\Rightarrow A^b$  is basic.

## Morita equivalence

Recall:  $\Lambda$  ring.  $M \in \text{mod } \Lambda$  is a left  $\text{End}_\Lambda(M)$ -module via  
 $f \cdot a \stackrel{\text{def}}{=} f(a) \Rightarrow$  right  $\text{End}_\Lambda(M)^{\text{op}}$ -module  
 $\forall a \in M$   $\Gamma :=$   
 $\Rightarrow M$  is a  $\Lambda$ - $\Gamma$ -bimodule.

Now, if  $X \in \text{mod } \Lambda \Rightarrow \text{Hom}_\Lambda(M, X)$  is a left  $\Gamma$ -module via  
 $(g \cdot f)(a) \stackrel{\text{def}}{=} f(g(a)) \quad \forall a \in M, g \in \Gamma$   
 $f \in \text{Hom}_\Lambda(M, X)$

$\Rightarrow$  Have a functor  $\text{Hom}_\Lambda(M, -) : \text{mod } \Lambda \rightarrow \text{mod } \Gamma$

Goal: Take  $M = \Lambda e_1$ . Then,  $\text{mod } \Lambda$  &  $\text{mod } \Lambda^b$  are Morita equivalent, that is, there exists an equivalence of categories

$$\text{Hom}_\Lambda(\Lambda e_1, -) : \text{mod } \Lambda \xrightarrow{\sim} \text{mod } \Lambda^b$$

"End( $\Lambda e_1$ )"  
" $\Gamma$ "

Lemma:  $M \in \text{mod } \Lambda$ , Artinian  $k$ -algebra.

Define the category  $\text{add } M = \{Z \mid \exists Y \text{ s.t. } Z \oplus Y \cong M^t \text{ for some } t \geq 1\}$

Let  $\Gamma = \text{End}_\Lambda(M)^{\text{op}}$ . Then

- (projectivization)
- $X \in \text{add } M \Rightarrow \text{Hom}_\Lambda(M, X)$  is a projective  $\Gamma$ -module
  - $e_M := \text{Hom}_\Lambda(M, -) : \text{add } M \rightarrow \text{proj } \Gamma$   
is an equivalence of categories.  
 $\hookrightarrow$  cat. of finitely generated projective  $\Gamma$ -module

Proof a)  $X \in \text{add } M \Rightarrow \exists n \text{ s.t. } M^n \cong X \oplus Y$  for some  $Y$ .

$$\Rightarrow \text{Hom}_A(M, M^n) \cong \text{Hom}_A(M, M)^n \cong \Gamma^n$$

$$\text{Hom}_A(M, X \oplus Y) \cong \text{Hom}_A(M, X) \oplus \text{Hom}_A(M, Y)$$

$\Rightarrow \text{Hom}_A(M, X)$  is a direct summand of  $\Gamma^n$   
so it is projective.

b) Exo  $\text{Hom}_A(M, -)$  is fully faithful, i.e.

$$\begin{array}{ccc} \text{Hom}_A(Z, X) & \xrightarrow{\sim} & \text{Hom}_\Gamma(\text{Hom}_A(M, Z), \text{Hom}_A(M, X)) \\ f \longmapsto & & (g \longmapsto g \circ f) \end{array}$$

Three cases: ①  $Z = M$  ②  $Z = M^n$  ③  $Z \in \text{add } M$ .

②  $\text{Hom}_A(M, -)$  is dense

Let  $P \in \text{proj } \Gamma$ .  $\exists n \geq 1$  s.t.  $\Gamma^n \cong P \oplus Q$   
&  $Q$

Let  $f': P \oplus Q \rightarrow P \oplus Q \Rightarrow f'$  is an idempotent  
 $(p, q) \mapsto (0, q)$  &  $\ker f' = P$

$$\begin{array}{ccc} \text{Also, } e_M: \text{Hom}_A(M^n, M^n) & \xrightarrow{\sim} & \text{Hom}_\Gamma(e_M M^n, e_M M^n) \\ & \cong & \text{Hom}_\Gamma(P^n, P^n) \\ & & \downarrow \sigma g \sigma^{-1} \end{array}$$

Let  $\psi^{-1} f' \psi \in \text{Hom}_{\mathcal{M}}(\Gamma^n, \Gamma^n)$

Choose  $u: M^n \rightarrow M^n$  s.t.  $\psi e_{\mathcal{M}}(u) = \psi^{-1} f' \psi: \Gamma^n \rightarrow \Gamma^n$

Let  $f = \psi^{-1} f' \psi \Rightarrow f^2 = (f')^2 = f'$

Have an exact sequence

$$0 \rightarrow \ker u \rightarrow M^n \xrightarrow{u} M^n$$

$$\rightsquigarrow 0 \rightarrow e_{\mathcal{M}}(\ker u) \rightarrow e_{\mathcal{M}}(M^n) \xrightarrow{e_{\mathcal{M}}(u)} e_{\mathcal{M}}(M^n)$$

$$\begin{array}{ccccccc} & & \uparrow & \text{EXO} & \uparrow & \sigma & \uparrow \sigma \\ & & \uparrow & \leftarrow & \uparrow & \sigma & \uparrow \sigma \\ 0 & \rightarrow & \ker f & \rightarrow & \Gamma^n & \xrightarrow{\psi e_{\mathcal{M}}(u) = f} & \Gamma^n \\ & & \downarrow \psi & \text{EXO} & \downarrow \psi & \sigma & \downarrow \psi \\ 0 & \rightarrow & P & \rightarrow & P \oplus Q & \xrightarrow{f'} & P \oplus Q \end{array}$$

$$\Rightarrow P \cong e_{\mathcal{M}}(\ker u)$$

Claim:  $\ker u \in \text{add } M$ . In fact,

$$\begin{aligned} f &= \psi e_{\mathcal{M}}(u) = f^2 = (\psi e_{\mathcal{M}}(u))^2 \\ &= \psi (e_{\mathcal{M}}(u)^2) = \psi (e_{\mathcal{M}}(u^2)) \end{aligned}$$

$\Rightarrow u = u^2$  since  $\psi$  &  $e_{\mathcal{M}}$  are iso.

$\Rightarrow M^n = \ker u \oplus \text{Im } u \Rightarrow \ker u \in \text{add } M$

$\Rightarrow P \cong e_{\mathcal{M}}(\ker u) \in \text{add } M \Rightarrow e_{\mathcal{M}}$  is dense.  $\square$



②  $\text{Hom}_1(\lambda_{e_1}, -)$  is fully faithful.

Let  $X, Y \in \text{mod } \Lambda$ . WTS  $\text{Hom}_1(X, Y) \xrightarrow{\cong} \text{Hom}_{\Lambda}(\text{Hom}_1(\lambda_{e_1}, X), \text{Hom}_1(\lambda_{e_1}, Y))$

$\begin{matrix} e_{e_1}(X) \\ \parallel \\ \text{Hom}_1(\lambda_{e_1}, X) \end{matrix}$ 
 $\begin{matrix} e_{e_1}(Y) \\ \parallel \\ \text{Hom}_1(\lambda_{e_1}, Y) \end{matrix}$

Take a projective presentation of  $X$ :

$\eta': Q_1' \rightarrow Q_0' \xrightarrow{u} X \rightarrow 0$  exact with  $Q_0', Q_1' \in \text{add}(\lambda_{e_1}) = \text{proj } \Lambda$

$\hookrightarrow$  since projective  $\Lambda$ -module appears as direct summand of  $\lambda_{e_1}$ .

Apply  $\text{Hom}_1(\lambda_{e_1}, -)$  to get

$\eta: \text{Hom}_1(\lambda_{e_1}, Q_1') \rightarrow \text{Hom}_1(\lambda_{e_1}, Q_0') \xrightarrow{u^*} \text{Hom}_1(\lambda_{e_1}, X) \rightarrow 0$

$f \longmapsto u \circ f$

exact since  $\lambda_{e_1}$  projective (Exo)

Consider two exact sequences

$\text{Hom}_1(\eta', Y): 0 \rightarrow \text{Hom}_1(X, Y) \rightarrow \text{Hom}_1(Q_0', Y) \rightarrow \text{Hom}_1(Q_1', Y)$

$\downarrow$  what we wanted to show.

$\swarrow$  Exo

$\downarrow$  by 2 (\*)

$\text{Hom}_{\Lambda}(-, e_{e_1}(Y)): 0 \rightarrow \text{Hom}_{\Lambda}(e_{e_1}(X), e_{e_1}(Y)) \rightarrow \text{Hom}_{\Lambda}(e_{e_1}(Q_0'), e_{e_1}(Y)) \rightarrow \text{Hom}_{\Lambda}(e_{e_1}(Q_1'), e_{e_1}(Y))$

□

Theorem:  $\Lambda$  f.d.  $k$ -algebra,  $k$  alg. closed

- a)  $\exists$  basic algebra  $\Lambda^b$  s.t.  $\Lambda \cong_m \Lambda^b$
- b) IF  $\Lambda$  is basic, then there exists  $\overset{\text{Morita equivalent}}{\text{a quiver}}$   $Q$  and an admissible  $I$  s.t.  $\Lambda \cong kQ/I$

Proof b)  $\Lambda$  basic  $\Rightarrow \Lambda/\mathfrak{r} \cong k^s$  for some  $s \geq 1$

Get a cspoi  $f_i \in k^s, f_i = (0, \dots, 0, 1, 0, \dots, 0)$   
 $\uparrow$  position  $i$

Vertices

Exo We can lift these to a cspoi in  $\Lambda$   
 $\{v_i\}_{i=1}^s$

Arrows

Now choose a basis  $B_{ji} = \{a_{ji}^l\}_{l=1}^{\dim B_{ji}}$  of the  
v.s.  $v_j \otimes_{\mathfrak{r}} v_i$

lift these to elements  $\tilde{B}_{ji} = \{\tilde{a}_{ji}^l\}_{l=1}^{\dim B_{ji}}$   
in  $v_j \otimes v_i$ .

Define the quiver  $Q$  by  $Q_0 = \{i\}_{i=1}^s$   
 $Q_1 = \{\alpha_{ij}^l : i \rightarrow j \mid l=1, \dots, \dim v_j \otimes v_i\}$

Define an homomorphism of  $k$ -algebras

$$\varphi : kQ \longrightarrow \Lambda$$

$$\begin{matrix} e_i & \longmapsto & v_i \\ \alpha_{ij}^l & \longmapsto & \tilde{a}_{ji}^l \end{matrix}$$

We show ①  $\ker \varphi$  is admissible  
②  $\varphi$  surjective

$$\Rightarrow \Lambda \cong kQ / \ker \varphi$$

① ker  $\psi$  is admissible

Suppose  $x \in \ker \psi$ .

$$x = \sum_{i \in Q_0} \gamma_i(0) e_i + \sum_{r, s \in K} \gamma_{r, s}(1) \alpha_{r, s}^p + \text{longer paths}$$

WTS  $x \in J^2$  i.e.  $\parallel 0$ .

$$\psi(x) = 0 \Rightarrow \psi(x) + \underline{r} = 0$$

$$\sum_{i \in Q_0} \gamma_i(0) v_i + \underbrace{\sum_{r, s \in K} \gamma_{r, s}(1) \tilde{\alpha}_{r, s}^p}_{\in \underline{r}} + \underline{r}$$

$$\Rightarrow \sum_{i \in Q_0} \gamma_i(0) v_i \in \underline{r}$$

$$\Rightarrow \sum_{i \in Q_0} \gamma_i(0) f_i = 0 \text{ in } \Lambda / \underline{r} = k^s$$

Basis of  $k^s$

$$\Rightarrow \gamma_i(0) = 0 \quad \forall i = 1, \dots, s$$

$$\text{Now } \psi(x) = 0 \Rightarrow \psi(x) + \underline{r}^2 = 0$$

$$\sum_{r, s, t} \gamma_{r, s, t} \tilde{\alpha}_{r, s, t}^p + \underline{r}^2$$

$$\Rightarrow \sum_{r, s, t} \gamma_{r, s, t} \tilde{\alpha}_{r, s, t}^p \in \underline{r}^2$$

$$\Rightarrow \sum_{r, s, t} \gamma_{r, s, t} \tilde{\alpha}_{r, s, t}^p = 0 \text{ in } \underline{r}^2$$

Basis of  $\underline{r}^2$

$$\Rightarrow \gamma_{r, s, t}(1) = 0 \quad \forall r, s, t$$

$$\Rightarrow x \in J^2 \Rightarrow \ker \psi \subseteq J^2.$$

⑦

Now wts  $J^m \subseteq \ker \varphi$  for some  $m$

Have  $r^m = (0)$  for some  $m$  since  $\lambda$  artinian  
&  $x_{rs}^e \mapsto \tilde{a}_{rs}^e \in \underline{r}$

$\Rightarrow$  {all paths of length  $\geq m$ }  $\subseteq \ker \varphi$

$\Rightarrow J^m \subseteq \ker \varphi \Rightarrow \ker \varphi$  is admissible.

②  $\varphi$  is surjective

Let  $\lambda \in \lambda$ .  $\Rightarrow \lambda \sum \delta_i(0) v_i \in \underline{r}$  for some  
 $\delta_i(0) \in k$   
 $i=1, \dots, r$

since  $\{\bar{v}_i = v_i + \mathfrak{m}\}_{i=1}^r$  is a basis of  $\lambda/\mathfrak{m}$

Have  $\underline{r}$  is generated by  $\{\tilde{a}_{ji}^e\} \in \text{Im } \varphi$

$\Rightarrow \lambda \sum \delta_i(0) v_i = \sum \delta_{rs}^{(1)} \tilde{a}_{rs}^e + \text{longer paths}$

$\Rightarrow \lambda \in \text{Im } \varphi \Rightarrow \varphi$  is surjective.

$$\Rightarrow \boxed{k[\mathcal{Q}] / k[\ker \varphi] \cong \lambda} \quad \square$$

1. The first part of the paper is devoted to the study of the

properties of the solutions of the system of equations

in the case of a constant coefficient

of the system of equations

with a constant coefficient

of the system of equations

with a constant coefficient

of the system of equations

with a constant coefficient

# Solutions to Exo session 10

#1a)  $\Rightarrow \Lambda$  basic  $\Leftrightarrow \Lambda/\mathfrak{r} = k \times \dots \times k$   
 $\Rightarrow$  every simple module is 1-dim'l

By induction on  $\ell(M)$ :

- $\ell(M) = 1 \Rightarrow M$  simple  $\Rightarrow \dim_k M = 1 = \ell(M) \checkmark$
- $\ell(M) > 1 \Rightarrow \exists N \neq M$
- $\Rightarrow 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$  s.e.s
- $\Rightarrow \ell(M) = \ell(N) + \ell(M/N)$
- $\dim M = \dim N + \dim M/N$   $\checkmark$   
"  $\leftarrow$  " ind " ind.

$\Leftrightarrow \ell(M) = \dim_k M \quad \forall$  f.g.  $M$   
 $\Rightarrow$  Every simple module is 1-dim'l.

$\Lambda/\mathfrak{r}$  semisimple  $\Rightarrow \Lambda/\mathfrak{r} \cong \prod_{i=1}^{\ell} M_{n_i}(k)$  (Wedderburn-Artin)  
 $\uparrow$  alg. closed  
 $\cong \bigoplus_{i=1}^{\ell} \underbrace{\text{top } \ell e_i}_{\text{simple } e}$

$\Rightarrow M_{n_i}(k) \cong \underbrace{\text{top } \ell e_i}_{1\text{-dim'l}} \Rightarrow \Lambda/\mathfrak{r} \cong \prod_{i=1}^{\ell} k.$   
(KRS)

b)  $M_n(k)$  semisimple  $\Rightarrow$  Every module is semisimple  
 $\Rightarrow$  indecomposable  $\Leftrightarrow$  simple  
 $\Rightarrow M$  indec  $\Rightarrow \ell(M) = 1.$

But  $M_n(k) \cong \bigoplus_{i=1}^n M_n(k)e_{ii}$  1 at position (ii)

$M_n(k)e_{ii} = \begin{pmatrix} 0 & & \\ & k & \\ & & 0 \end{pmatrix}$  these are all isomorphic simple modules.  
 $\Rightarrow \dim_k M_n(k)e_{ii} = n.$

#2  $I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ k & k & k & 0 \end{pmatrix}$  is nilpotent &  $\Lambda/I$  semisimple

$\Rightarrow I = \text{rad } \Lambda$  &  $\Lambda/I = k \times k \times k \times k$

$\Rightarrow \Lambda$  is basic

Alternatively  $\{e_{11}, e_{22}, e_{33}, e_{44}\}$  cs poi

$$\& \Lambda \cong \bigoplus_{i=1}^4 \Lambda e_{ii} \cong \begin{pmatrix} k & & & \\ & k & & \\ & & k & \\ & & & k \end{pmatrix} \oplus \begin{pmatrix} k & & & \\ & k & & \\ & & k & \\ & & & k \end{pmatrix} \oplus \begin{pmatrix} k & & & \\ & k & & \\ & & k & \\ & & & k \end{pmatrix} \oplus \begin{pmatrix} k & & & \\ & k & & \\ & & k & \\ & & & k \end{pmatrix}$$

& these are pairwise non-isomorphic.

#3a)  $\Lambda' \subset \Lambda \Rightarrow \text{rad } \Lambda = \text{rad } \Lambda' \cdot \Lambda$

$$\Rightarrow \Lambda' / \text{rad } \Lambda' \subset \Lambda / (\text{rad } \Lambda' \cdot \Lambda) = k \times \dots \times k$$

Since  $\Lambda' / \text{rad } \Lambda'$  semisimple &  $k$  alg. closed, have

$$\Lambda' / \text{rad } \Lambda' = M_{n_1}(k) \times \dots \times M_{n_j}(k) \subset k \times \dots \times k$$

$\Rightarrow n_i = 1$  for  $i=1, \dots, j$  &  $\Lambda'$  is basic.

b) similar.