Galois theory - Problem Set 6

To be solved on Thursday 02.05

Chapter 18.3

Problem 1. Is the polynomial $f(x) = x^5 - x^4 - x + 1$ solvable by radicals?

Problem 2. (Exercise 18.3.1 in the book.) Show that the following polynomials are not solvable by radicals over $\mathbb{Q}$:

(a) $x^5 - 9x + 3$.

(b) $2x^5 - 5x^4 + 5$.

(c) $x^5 - 8x + 6$.

(d) $x^5 - 4x + 2$.

Problem 3. (a) Let G be an abelian group. Show that G is simple if and only if it is cyclic of prime order.

(b) Show that finite abelian groups are solvable. (*Hint*: you may use Problem 13(d), that is that the direct product of two solvable groups is solvable.)

Chapter 18.5

Problem 4. Find for which $n \in \mathbb{Z}_{\geq 1}$ is $\sqrt[n]{2}$ constructible.

Problem 5. (Exercise 18.3.4 in the book.) Prove that the regular 17-gon is constructible with ruler and compass.

Problem 6. (Exercise 18.5.1 in the book.) Show that the angle $\frac{2\pi}{5}$ can be trisected using ruler and compass.

Problem 7. (Exercise 18.3.2 in the book.) Show that it is impossible to construct a regular 9-gon or 7-gon using ruler and compass.

Problem 8. (Exercise 18.3.3 in the book.) Show that it is possible to trisect $54°$ using ruler and compass.

Problem 9. Let $\mathbb{K}$ be the set of constructible numbers and $\mathbb{A}$ be the set of algebraic numbers.

(a) Does $\mathbb{K} \subseteq \mathbb{A}$ hold?

(b) Does $\mathbb{A} \subseteq \mathbb{K}$ hold?

Problem 10. (Exercise 18.3.5 in the book.) Find which of the following numbers are constructible:

(i) $\sqrt{3} + 1$.

(ii) $\pi^2 + 1$.

(iii) $\sqrt{\sqrt{3} - 1} + 1$.

(iv) $\sqrt[3]{2} + 1$.

(v) $\sqrt[4]{\sqrt{2}+\sqrt{5}}$.

Problem 11. Let L be a line and P be a point in $\mathbb{C}$.

(a) Using ruler and compass, show that we may draw the line that goes through P and is perpendicular to L.

(b) Using ruler and compass, show that we may draw the line that goes through P and is parallel to L.

Problem 12. Let $0 \leq \theta < 2\pi$. We say that an angle of measure θ is *constructible* if there exist constructible $O, P, Q \in \mathbb{C}$ such that the segments (OP) and (OQ) form an angle of measure θ. Show that the following are equivalent.

(a) An angle of measure θ is constructible.

(b) The number $\cos(\theta)$ is constructible.

(c) The number $\sin(\theta)$ is constructible.

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 13. (Chapter 18.3) Assume all groups in this exercise are finite.

(a) Let G be a solvable group and $H < G$ be a subgroup. Show that H is solvable.

(b) Let G be a solvable group and $H \triangleleft G$ be a normal subgroup. Show that G/H is solvable.

(c) Let G be a group and $H \triangleleft G$ be a normal subgroup. Show that if H and G/H are solvable, then G is also solvable.

(d) Show that if G_1 and G_2 are solvable groups, then $G_1 \times G_2$ is a solvable group.

Problem 14. (Chapter 18.3) Show that the symmetric group S_n is solvable if and only if $n \leq 4$. (*Hint:* consider the cases $n = 1, 2, 3, 4$ and $n \geq 5$ separately.)

Problem 15. (Chapter 18.5) This problem aims to demonstrate that the last sentence of Theorem 18.5.9 in the book is wrong. That is, we show that there exists α such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ but α is not constructible.

Assume that $z \in \mathbb{K}$ is a constructible number.

(a) Show that there exists a normal field extension $\mathbb{Q} \subseteq N$ such that $z \in N$ and $[N : \mathbb{Q}] = 2^n$ for some $n \geq 0$. (*Hint:* Use Theorem 17.9 to obtain a sequence of field extensions and use induction to show that each of the fields in that sequence is included in another field as part of a normal extension of degree some power of 2.)

(b) Show that the splitting field of the minimal polynomial of z over $\mathbb{Q}$ has degree 2^n for some $n \geq 0$.

(c) Show that the Galois group of the minimal polynomial of z over $\mathbb{Q}$ has degree 2^n for some $n \geq 0$.

(d) Assume that we are given a monic polynomial $f(x) \in \mathbb{Q}[x]$ which is irreducible over $\mathbb{Q}$, of degree 4 and such that the Galois group of $f(x)$ is S_4[1]. Let α be a root of $f(x)$. Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ but α is not constructible.

[1] Example: $x^4 - x - 1$ is one such polynomial, although with our tools it is not easy to see that this polynomial satisfies these requirements. One needs the notions of *resolvent* and *discriminant* to show this.