

Galois theory - Problem Set 2

To be solved on Thursday 15.02

Chapter 15.3

Problem 1. (Exercise 15.3.2 in the book.) Prove that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over \mathbb{Q} . Find the degree of

- (a) $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} .
- (b) $\mathbb{Q}(\sqrt{3})$ over \mathbb{Q} .
- (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over \mathbb{Q} .
- (d) $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over \mathbb{Q} .

Problem 2. (Exercise 15.3.4 in the book) Find a suitable number α such that

- (a) $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\alpha)$.
- (b) $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\alpha)$.

Problem 3. (Exam May 2013, Problem 3.)

- (a) Let α be an algebraic number over the field F such that $[F(\alpha) : F]$ is an odd number. Show that this implies that $F(\alpha^2) = F(\alpha)$.
- (b) Give an example to show that the converse implication is not true (Hint: Cyclotomic extensions.)

Problem 4. (Exam June 2015, Problem 3.) Let $F \subseteq E$ be a field extension of degree $[E : F] = n$.

- (a) Show that if n is a prime number, then there is no proper intermediate field between E and F (that is, no field K with $F \subseteq K \subseteq E$ and $E \neq K \neq F$). Deduce that if $\alpha \in E \setminus F$, then the minimal polynomial of α in $F[x]$ has degree n .
- (b) Let $E = F(\alpha, \beta)$, where α has minimal polynomial in $F[x]$ of degree d_1 , and β has minimal polynomial in $F[x]$ of degree d_2 . Show that if d_1 and d_2 are coprime (i.e. $\gcd(d_1, d_2) = 1$), then $[E : F] = d_1 d_2$.
- (c) Give an example where α and β are as in (b), and such that $\alpha\beta$ has minimal polynomial in $F[x]$ of degree d_1 or d_2 . (Hint: consider $F = \mathbb{Q}$ with $\alpha = \sqrt[3]{2}$ and β a suitable root of unity.)

Problem 5. (Exercise 15.4.8 in the book.) Let F be a field and let $n \geq 1$. Let $f(x) = x^n - \alpha \in F[x]$ be an irreducible polynomial over F and let $b \in K$ be a root of $f(x)$, where $\overline{F} \subseteq K$ is a field extension. If m is a positive integer such that $m \mid n$, find the degree of the minimal polynomial of b^m over F .

Chapter 15.4

Problem 6. (Exam June 2014, Problem 3.) Let $f(x) \in F[x]$ be a nonzero polynomial over the field F with various properties as described below. Let $\alpha \in \overline{F}$, where \overline{F} denotes the algebraic closure of F .

- (a) Let $f(\alpha) = 0$. Assume that whenever $g(\alpha) = 0$ for some nonzero $g(x) \in F[x]$, then $\deg(f) \leq \deg(g)$. Show that $f(x)$ is irreducible over F .
- (b) Show the converse of (a), that is: Assume $f(x)$ is irreducible over F and $f(\alpha) = 0$. Let $g(\alpha) = 0$ for some nonzero $g(x) \in F[x]$. Show that $\deg(f) \leq \deg(g)$.

Problem 7. (Exam August 2013, Problem 4.) Let $f(x) \in F[x]$ be an irreducible polynomial of prime degree p over the field F , with $\text{char}(F) = 0$ (**Warning: I don't think the characteristic of F plays a role.**). Let $K = F(\alpha)$, where α is a root of an irreducible polynomial $g(x) \in F[x]$ of prime degree q over the field F . Assume $f(x)$ is reducible in $K[x]$. Show that $p = q$.

Chapter 16.1

Problem 8. (Exercise 16.1.1 in the book.) Construct splitting fields K over \mathbb{Q} for the polynomial $f(x)$ and find the degree $[K : \mathbb{Q}]$ where $f(x)$ is

- (a) $x^3 - 1$.
- (b) $x^4 + 1$.
- (c) $x^6 - 1$.
- (d) $(x^2 - 2)(x^3 - 3)$.

Problem 9. (Exercise 16.1.2 in the book.) Construct a splitting field for $x^3 + x + 1 \in \mathbb{Z}_2[x]$ and list all its elements.

Problem 10. (Exercise 16.1.5 in the book.) Let E be the splitting field of a polynomial of degree n over a field F . Show that $[E : F] \leq n!$.

Chapter 16.2

Problem 11. (Exercise 16.2.2 in the book.) Is $\mathbb{R} \subseteq \mathbb{R}(\sqrt{-5})$ a normal field extension?

Problem 12. (Exercise 16.2.3 in the book.) Let E be a normal extension of F and let K be a subfield of E containing F . Show that E is a normal extension over K . Give an example to show that K need not be a normal extension of F .

Problem 13. (Exercise 16.2.4 in the book.) Let $F = \mathbb{Q}(\sqrt{2})$ and $E = \mathbb{Q}(\sqrt[4]{2})$. Show that E is a normal extension of F , F is a normal extension of \mathbb{Q} , but E is not a normal extension of \mathbb{Q} .

Problem 14. (Exercise 16.2.6 in the book.) Let E_i , $i \in \Lambda$ be a family of normal extensions of a field F in some extension K of F . Show that $E := \bigcap_{i \in \Lambda} E_i$ is also a normal extension of F .

Problem 15. (Exam June 2014, Problem 5.)

- (a) Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}^+$. Find the minimal polynomial of α over \mathbb{Q} .
- (b) Show that $\mathbb{Q}(\alpha)$ is a normal extension of \mathbb{Q} . (Hint: Consider $\alpha\sqrt{2 - \sqrt{2}}$.)

Extra problems

Problem 16. (Chapter 16.1) Let $f(x) = x^3 + ax + b \in \mathbb{Q}[x]$. Let E be the splitting field of f . Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be the roots of f (not necessarily distinct).

- (a) Define $D = (\alpha_2 - \alpha_1)^2(\alpha_3 - \alpha_1)^2(\alpha_3 - \alpha_2)^2$. Show that $D = -(4a^3 + 27b^2)$.
- (b) Show that if $f(x)$ is reducible, then $[E : \mathbb{Q}] = 1$ or $[E : \mathbb{Q}] = 2$.
- (c) (Exercise 16.1.3 in the book.) Show that if $f(x)$ is irreducible and $\sqrt{D} \in \mathbb{Q}$, then $[E : \mathbb{Q}] = 3$.
- (d) (Exercise 16.1.4 in the book.) Show that if $f(x)$ is irreducible and $\sqrt{D} \notin \mathbb{Q}$, then $[E : \mathbb{Q}] = 6$.
- (e) (Exercise 16.1.8 in the book.) Show that over any field $K \supseteq \mathbb{Q}$ the polynomial $x^3 - 3x + 1$ is either irreducible or splits into linear factors.

Problem 17. (Chapter 16.1) Let E be the splitting field of a polynomial of degree n over a field F . Show that $[E : F]$ divides $n!$.