This generalizes to the following

<u>Theorem 6.5.</u> Let $\sigma: F \to L$ be an embedding of F into an algebraically closed field L. Let $F \subseteq E$ be an algebraic extension. Then there exists an embedding $\sigma^*: E \to L$ such that $\sigma^*|_F = \sigma$.

<u>Proof.</u> Let

$S = \{(K, \theta) \mid F \subseteq K \subseteq E$ field extensions, $\theta: K \to L$ embedding, $\theta|_F = \sigma\}$

Write $(K, \theta) \leq (K', \theta')$ if $K \subseteq K'$ and $\theta'|_K = \theta$. It is clear that $(S, \leq)$ is a partially ordered set. Moreover, $(F, \sigma) \in S$ so $S \neq \emptyset$. Let $\{(K_i, \theta_i)\}_{i \in I}$ be a chain in S (that is a totally ordered subset). We set $K = \bigcup_{i \in I} K_i$. Moreover, for $\alpha \in K$ we have $\alpha \in K_i$ for some $i \in I$ and we set $\theta(\alpha) = \theta_i(\alpha)$. Clearly $(K, \theta) \in S$ and (K, θ) is an upper bound for $\{(K_i, \theta_i)\}_{i \in I}$. By Zorn's Lemma there exists a maximal element $(M, \eta) \in S$. Assume towards a contradiction that $M \subsetneq E$. Let $\alpha \in E \setminus K$. By Lemma 6.4 there exists $\eta^*: M(\alpha) \to E$ such that $\eta^*|_K = \eta$. In particular, $(M, \eta) \leq (M(\alpha), \eta^*)$ and by maximality of (M, η) we conclude that $M = M(\alpha) \Rightarrow \alpha \in M$, a contradiction. Hence $E = M$ and $\sigma^* = \eta: K \to L$ is the required embedding. $\square$

<u>Theorem 6.6.</u> Let F be a field and K, K' be algebraic closures of F. Then $\exists$ F-isomorphism $\varphi: K \xrightarrow{\sim} K'$.

<u>Proof.</u> We have

$$\left.\begin{array}{l} F \subseteq K \Rightarrow \exists \text{ embedding } \sigma: F \to K, \sigma(\alpha) = \alpha \\ F \subseteq K' \text{ is an algebraic extension} \end{array}\right\} \xrightarrow{\text{Theorem 6.5.}} \exists \sigma^*: K' \to K \text{ such that } \sigma^*|_F = \sigma.$$

Then $\sigma^*(K') \cong K'$ is an algebraically closed subfield of K. Since $\sigma^*(F) = \sigma(F) = F$ it follows that $F \subseteq \sigma^*(K')$. Then

$$F \subseteq \sigma^*(K') \subseteq K$$

and $F \subseteq K$ is algebraic $\Rightarrow \sigma^*(K') \subseteq K$ is algebraic. But $\sigma^*(K')$ is algebraically closed and so $\sigma^*(K') = K \Rightarrow \sigma^*$ is an F-isomorphism. $\square$

Hence we have shown that if an algebraic closure exists, then it is unique up to isomorphism; we denote this algebraic closure by $\overline{F}$. Now we turn our attention to the existence of $\overline{F}$.

Theorem 6.7. $\exists$ an algebraically closed field K that contains F as a subfield.

Proof. (Artin) For a polynomial $f(x) \in F[x]$ with $\deg(f) \geq 1$, define a formal variable x_f. Consider the infinite polynomial ring

$$R = F[x_f]_{f(x) \in F[x],\, \deg(f) \geq 1}.$$

That is, R contains polynomials in finitely many of the infinitely many variables x_f (addition and multiplication is defined as usual). The degree of a monomial $a_{f_1 \cdots f_n} x_{f_1}^{m_1} \cdots x_{f_n}^{m_n} \in R$, $a_{f_1 \cdots f_n} \neq 0$, is defined to be $m_1 + \cdots + m_n$, and a degree of a general polynomial is defined to be the maximal degree of all monomials.

Define I to be the ideal in R generated by all polynomials $f(x_f) \in R$ where $f(x) \in F[x]$, $\deg(f) \geq 1$. Assume to a contradiction that $1 \in I$. Then

$$1 = g_1 \cdot f_1(x_{f_1}) + \cdots + g_n \cdot f_n(x_{f_n}) \qquad (*)$$

For some $g_i \in R$, $f_i(x) \in F[x]$, $\deg(f_i) \geq 1$. Each g_i has only a finite number of variables. Hence, we may reindex the variables so that $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m$ are all the variables occuring in $(*)$, and $x_i = x_{f_i}$. Then $(*)$ becomes

$$g_1(x_1,\dots,x_m)f_1(x_1)+\dots+g_n(x_1,\dots,x_m)f_n(x_n)=1. \quad (**)$$

There exists a field extension $F\subseteq E$ such that each of the polynomials $f_1,\dots,f_n$ has a root $\alpha_i\in E$ (exercise). Evaluating $(**)$ at $(x_1,\dots,x_n,x_{n+1},\dots,x_m)=(\alpha_1,\dots,\alpha_n,0,\dots,0)$ we obtain $0=1$, a contradiction. Hence $1\notin I$ and so $I\neq R$.

Since $I\neq R$, there exists a maximal ideal m of R such that $I\subseteq m\subsetneq R$ (exercise). Hence R/m is a field. Moreover, the constant polynomials in R correspond to F, and these do not belong to m since $m\neq R$ (except for the zero polynomial). Hence $F\subseteq R/m$ is a field extension. Moreover, by construction, we have for every $f(x)\in F[x]$ with $\deg(f)\geq 1$ that $\overline{x_f}=x_f+R/m\in R/m$ and

$$f(\overline{x_f})=f(x_f)+R/m=0$$

since $f(x_f)\in I\subseteq m$.

Now set $K_1=R/m$. We have

$$F\subseteq K_1$$

and every polynomial $f(x)\in F[x]$ with $\deg(f)\geq 1$ has a root in K_1. We may then continue inductively and build an infinite sequence

$$F=K_0\subseteq K_1\subseteq K_2\subseteq\dots\subseteq K_n\subseteq\dots$$

such that every polynomial $f(x)\in K_n[x]$ with $\deg(f)\geq 1$ has a root in K_{n+1}. Set $K=\bigcup_{i=0}^{\infty}K_i$. Let $f(x)\in K[x]$, $\deg(f)\geq 1$. Then $\exists$ n such that $f(x)\in K_n[x]$ and so f has a root in K_{n+1}. Since $K_{n+1}\subseteq K$, f has a root in K. Then K is algebraically closed by Theorem 6.3. $\square$

Note: Theorem 6.7 gives the existence of a field extension $F\subseteq K$ with K algebraically closed. In general, it is not necessary that K is an algebraic closure of F.

Lemma 6.8 (Problem 15.3.1) Let $F \subseteq K \subseteq E$ be field extensions such that $F \subseteq K$ is algebraic. Let $\alpha \in E$ be algebraic over K. Then α is algebraic over F.

Proof. Exercise.

Theorem 6.9. There exists an algebraic closure $\overline{F}$ of F.

Proof. By Theorem 6.7 there exists an algebraically closed field K containing F. Let

$$\overline{F} = \{\alpha \in K \mid \alpha \text{ is algebraic over } F\}.$$

By Corollary 5.9, $F \subseteq \overline{F} \subseteq K$ are field extensions and $F \subseteq \overline{F}$ is algebraic. Let $f(x) \in \overline{F}[x] \subseteq K[x]$. Since K is algebraically closed, $\exists$ a root $\alpha \in K$ of f. Then α is algebraic over $\overline{F}$ and so by Lemma 6.8 α is algebraic over F. Hence $\alpha \in \overline{F}$ and f has a root in $\overline{F}$. We conclude that $\overline{F}$ is algebraically closed and hence $\overline{F}$ is the algebraic closure of F. $\square$

Example 6.10. We will see that $\overline{\mathbb{C}} = \mathbb{C}$. On the other hand, $\mathbb{R}$ is not algebraically closed ($x^2+1 \in \mathbb{R}[x]$ has no root in $\mathbb{R}$). Hence $\mathbb{R} \subsetneq \overline{\mathbb{R}}$ and $[\overline{\mathbb{R}}:\mathbb{R}] \geq 2$. Following the proof of Theorem 6.9 we obtain $\mathbb{R} \subsetneq \overline{\mathbb{R}} \subseteq \mathbb{C}$ and so

$$2 = [\mathbb{C}:\mathbb{R}] = [\mathbb{C}:\overline{\mathbb{R}}]\cdot[\overline{\mathbb{R}}:\mathbb{R}] \geq 1 \cdot 2.$$

We conclude that $\overline{\mathbb{R}} = \mathbb{C}$. Notice that the same argument does not hold if we replace $\mathbb{R}$ by $\mathbb{Q}$ since $[\mathbb{C}:\mathbb{Q}] = \infty$ (Recall $\overline{\mathbb{Q}} = \{\text{algebraic numbers}\}$.)