Definition 5.6. A field extension $F \subseteq E$ is called finitely generated if $\exists\, \alpha_1, \dots, \alpha_n \in E$ such that $E = F(\alpha_1, \dots, \alpha_n)$.

Clearly a finite extension is finitely generated. The opposite is not true.

Example 5.7. Let $t \in \mathbb{R}$ be a transcedeytal number (e.g. $t = \pi$ or $t = e$). Then $\mathbb{Q} \subseteq \mathbb{Q}(t)$ is a finitely generated field extension, but is not finite since t is not algebraic.

However, we have the following.

Theorem 5.8. Let $F \subseteq E = F(\alpha_1, \dots, \alpha_n)$ be a finitely generated field extension such that α_i is algebraic over F for $1 \le i \le n$. Then it is a finite and, by Theorem 5.3, an algebraic extension.

Proof. We have

$$F \subseteq F(\alpha_1) \subseteq F(\alpha_1, \alpha_2) \subseteq \cdots \subseteq F(\alpha_1, \dots, \alpha_n) = E.$$

Moreover, α_i algebraic over $F \Rightarrow \alpha_i$ algebraic over $F(\alpha_1, \dots, \alpha_{i-1})$. Hence $[F(\alpha_1, \dots, \alpha_i) : F(\alpha_1, \dots, \alpha_{i-1})] < \infty$ $\forall\, 1 \le i \le n$. By Theorem 4.2 we obtain

$$[E : F] = [F(\alpha_1, \dots, \alpha_n) : F(\alpha_1, \dots, \alpha_{n-1})] \cdots [F(\alpha_1) : F] < \infty,$$

as required. □

Corollary 5.9. Let $F \subseteq E$ be a field extension. The set

$$K = \{\alpha \in E \mid \alpha \text{ algebraic over } F\}$$

is a subfield of E and an algebraic extension of F, called the algebraic closure of F in E.

Proof. It is enough to show that K is a subfield of E. If $\alpha, \beta \in K^*$,

then $F \subseteq F(\alpha,\beta)$ is algebraic by Theorem 5.8. and so $F(\alpha,\beta) \subseteq K$. But then $-\alpha, \alpha^{-1}, \alpha+\beta, \alpha-\beta \in K$ and so K is a subfield of E. $\square$

Example 5.10. Algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ $= \{\alpha \in \mathbb{C} \mid \exists f(x) \in \mathbb{Q}[x]$ with $f(\alpha)=0\}$
$= \{\alpha \in \mathbb{C} \mid \exists f(x) \in \mathbb{Z}[x]$ with $f(\alpha)=0\}$
$= \{$algebraic numbers$\}$

Definition 5.11. Let $F \subseteq K$, $F \subseteq L$ be field extensions. A field embedding $\sigma: K \longrightarrow L$ (i.e. nonzero ring homomorphism) is called an F-homomorphism or an embedding over F if $\sigma|_F = id_F$ that is, $\sigma(\alpha)=\alpha$ $\forall \alpha \in F$.

Theorem 5.12. Let $F \subseteq E$ be an algebraic field extension and $\sigma: E \longrightarrow E$ an F-homomorphism. Then σ is onto and hence an isomorphism.

Proof. Let $\alpha \in E$ and let $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ be the minimal polynomial of α over F. Let $S = \{r \in E \mid p(r)=0\}$. Since σ is an F-homomorphism, for every $r \in S$ we have

$$p(\sigma(r)) = a_0 + a_1\sigma(r) + \cdots + a_{n-1}\sigma(r)^{n-1} + \sigma(r)^n = \sigma(a_0) + \sigma(a_1)\sigma(r) + \cdots + \sigma(a_{n-1})\sigma(r)^{n-1} + \sigma(r)^n$$
$$= \sigma(a_0 + a_1 r + \cdots + a_{n-1}r^{n-1} + r^n) = \sigma(p(r)) = \sigma(0) = 0.$$

Hence σ maps S to S. Since σ is injective and S is a finite set, $\sigma|_S : S \longrightarrow S$ is a bijection. The claim follows since $\alpha \in S$ $\square$

Example 5.13. Theorem 5.12 hints at a connection between algebraic extensions $F \subseteq E$ and F-homomorphisms. We have $\mathbb{R} \subseteq \mathbb{R}(i) = \mathbb{C}$ and $[\mathbb{C}:\mathbb{R}] = \deg(x^2+1) = 2$. An $\mathbb{R}$-homomorphism $\sigma: \mathbb{C} \longrightarrow \mathbb{C}$ satisfies

$$\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$$

and so $\sigma(i) \in \{-i, i\}$. On the other hand, we have

$$\sigma(a+bi) = \sigma(a) + \sigma(b)\sigma(i) = a + b\sigma(i)$$

and so there exist exactly two $\mathbb{R}$-homomorphisms of $\mathbb{C}$:

$$\sigma(a+bi) = a+bi \quad \text{and} \quad \sigma(a+bi) = a-bi \ (= \overline{a+bi}).$$

These correspond to the roots $i, -i$ of the minimal polynomial x^2+1 of i over $\mathbb{R}$.