Galois theory - Problem Set 2

To be solved on Friday 10.02

Problem 1. (Exercise 15.3.2 in the book.) Prove that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over $\mathbb{Q}$. Find the degree of

(a) $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

(b) $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$.

(c) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

(d) $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$.

Problem 2. (Exercise 15.3.4 in the book) Find a suitable number α such that

(a) $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\alpha)$.

(b) $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\alpha)$.

Problem 3. (Exam June 2014, Problem 1.)

(a) Write down the irreducible polynomials over $\mathbb{Z}_2$ of degrees two and three, respectively.

(b) How many irreducible polynomials of degree four are there over $\mathbb{Z}_2$?

Problem 4. (Exam June 2014, Problem 3.) Let $f(x) \in F[x]$ be a nonzero polynomial over the field F with various properties as described below. Let $\alpha \in \overline{F}$, where $\overline{F}$ denotes the algebraic closure of F.

(a) Let $f(\alpha) = 0$. Assume that whenever $g(\alpha) = 0$ for some nonzero $g(x) \in F[x]$, then $\deg(f) \leq \deg(g)$. Show that $f(x)$ is irreducible over F.

(b) Show the converse of (a), that is: Assume $f(x)$ is irreducible over F and $f(\alpha) = 0$. Let $g(\alpha) = 0$ for some nonzero $g(x) \in F[x]$. Show that $\deg(f) \leq \deg(g)$.

Problem 5. (Exam May 2013, Problem 3.)

(a) Let α be an algebraic number over the field F such that $[F(\alpha) : F]$ is an odd number. Show that this implies that $F(\alpha^2) = F(\alpha)$.

(b) Give an example to show that the converse implication is not true (Hint: Cyclotomic extensions.)

Problem 6. (Exam June 2015, Problem 3.) Let $F \subseteq E$ be a field extension of degree $[E : F] = n$.

(a) Show that if n is a prime number, then there is no proper intermediate field between E and F (that is, no field K with $F \subseteq K \subseteq E$ and $E \neq K \neq F$). Deduce that if $\alpha \in E \setminus F$, then the minimal polynomial of α in $F[x]$ has degree n.

(b) Let $E = F(\alpha, \beta)$, where α has minimal polynomial in $F[x]$ of degree d_1, and β has minimal polynomial in $F[x]$ of degree d_2. Show that if d_1 and d_2 are coprime (i.e. $\gcd(d_1, d_2) = 1$), then $[E : F] = d_1 d_2$.

(c) Give an example where α and β are as in (b), and such that $\alpha\beta$ has minimal polynomial in $F[x]$ of degree d_1 or d_2. (Hint: consider $F = \mathbb{Q}$ with $\alpha = \sqrt[3]{2}$ and β a suitable root of unity.)

Problem 7. (Exercise 15.4.8 in the book.) Let F be a field and let $n \geq 1$. Let $f(x) = x^n - \alpha \in F[x]$ be an irreducible polynomial over F and let $b \in K$ be a root of f, where $F \subseteq K$ is a field extension. If m is a positive integer such that $m \mid n$, find the degree of the minimal polynomial of b^m over F.

Problem 8. (Exam August 2013, Problem 4.) Let $f(x) \in F[x]$ be an irreducible polynomial of prime degree p over the field F, with $\mathrm{char}(F) = 0$ (**Warning: I don't think the characteristic of F plays a role.**). Let $K = F(\alpha)$, where α is a root of an irreducible polynomial $g(x) \in F[x]$ of prime degree q over the field F. Assume $f(x)$ is reducible in $K[x]$. Show that $p = q$.

Problem 9. (**Warning: Needs field of fractions.**) (Exercise 15.4.10 in the book.) Give an example of a field E containing a proper subfield K such that E is embeddable in K and $[E : K]$ is finite.

Problem 10. (Exercise 16.1.1 in the book.) Construct splitting fields K over $\mathbb{Q}$ for the polynomial $f(x)$ and find the degree $[K : \mathbb{Q}]$ where $f(x)$ is

(a) $x^3 - 1$.

(b) $x^4 + 1$.

(c) $x^6 - 1$.

(d) $(x^2 - 2)(x^3 - 3)$.

Problem 11. (Exam June 2014, Problem 7.) Show that $\sqrt{2} + \sqrt[3]{3} \notin \mathbb{Q}$. (Hint: Consider an appropriate field extension of $\mathbb{Q}$.)

Problem 12. (Exercise 16.1.2 in the book.) Construct a splitting field for $x^3 + x + 1 \in \mathbb{Z}_2[x]$ and list all its elements.

Problem 13. (Exercise 16.1.5 in the book.) Let E be the spliting field of a polynomial of degree n over a field F. Show that $[E : F] \leq n!$.

Problem 14. Let $f(x) = x^3 + ax + b \in \mathbb{Q}[x]$. Let E be the splitting field of f. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be the roots of f (not necessarily distinct).

(a) Define $D = (\alpha_2 - \alpha_1)^2(\alpha_3 - \alpha_1)^2(\alpha_3 - \alpha_2)^2$. Show that $D = -(4a^3 + 27b^2)$.

(b) Show that if $f(x)$ is reducible, then $[E : \mathbb{Q}] = 1$ or $[E : \mathbb{Q}] = 2$.

(c) (Exercise 16.1.3 in the book.) Show that if $f(x)$ is irreducible and $\sqrt{D} \in \mathbb{Q}$, then $[E : \mathbb{Q}] = 3$.

(d) (Exercise 16.1.4 in the book.) Show that if $f(x)$ is irreducible and $\sqrt{D} \notin \mathbb{Q}$, then $[E : \mathbb{Q}] = 6$.

Problem 15. (Exercise 16.1.8 in the book.) Show that over any field $K \supseteq \mathbb{Q}$ the polynomial $x^3 - 3x + 1$ is either irreducible or splits into linear factors.

Problem 16. (Exercise 16.2.2 in the book.) Is $\mathbb{R} \subseteq \mathbb{R}(\sqrt{-5})$ a normal field extension?

Problem 17. (Exercise 16.2.3 in the book.) Let E be a normal extension of F and let K be a subfield of E containing F. Show that E is a normal extension over K. Give an example to show that K need not be a normal extension of F.

Problem 18. (Exercise 16.2.4 in the book.) Let $F = \mathbb{Q}(\sqrt{2})$ and $E = \mathbb{Q}(\sqrt[4]{2})$. Show that E is a normal extension of F, F is a normal extension of $\mathbb{Q}$, but E is not a normal extension of $\mathbb{Q}$.

Problem 19. (Exercise 16.2.6 in the book.) Let E_i, $i \in \Lambda$ be a family of normal extensions of a field F in some extension K of F. Show that $E := \bigcap_{i \in \Lambda} E_i$ is also a normal extension of F.

Problem 20. (Exam June 2014, Problem 5.)

(a) Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}^+$. Find the minimal polynomial of α over $\mathbb{Q}$.

(b) Show that $\mathbb{Q}(\alpha)$ is a normal extension of $\mathbb{Q}$. (Hint: Consider $\alpha\sqrt{2 - \sqrt{2}}$.)