Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam: Christian Skau, phone 91755

Exam in MA3202 Galoisteori

English
Thursday 16. May 2013
Time: 09.00 - 13:00
Permitted aids: None

Results: 10. June 2013

Problem 1

a) Let E be the splitting field of $f(x) = x^{14} - 1$ over $\mathbf{Q}$. Show that the Galois group $G = G(E|\mathbf{Q})$ is abelian.

b) Let $\tilde{E}$ be the splitting field of $g(x) = x^7 + 1$ over $\mathbf{Q}$. Show that the Galois group $\tilde{G} = G(\tilde{E}|\mathbf{Q})$ is abelian.

Problem 2

a) List all fields K such that $\mathbf{Q} \subseteq K \subseteq \mathbf{Q}(\omega, \sqrt[3]{2})$, where $\sqrt[3]{2} \in \mathbf{R}, \omega = e^{\frac{4\pi i}{3}}$. How many of these are normal extensions of $\mathbf{Q}$? Give explanation.

b) Which of the following fields are isomorphic? Give reasons by referring to theorems.

(i) $\mathbf{Q}(\sqrt[4]{2})$ and $\mathbf{Q}(i\sqrt[4]{2})$

(ii) $\mathbf{Q}(\sqrt[4]{1+\sqrt{3}})$ and $\mathbf{Q}(\sqrt[4]{1-\sqrt{3}})$

(iii) $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$.

Problem 3

a) Let α be an algebraic number over the field F such that $[F(\alpha) : F]$ is an odd number. Show that this implies that $F(\alpha^2) = F(\alpha)$.

b) Give an example to show that the converse implication is not true.
(Hint: Cyclotomic extensions.)

Problem 4

a) Show that $\mathbf{Z}[\sqrt{-10}]$ is not a euclidean domain.

b) Show that $\mathbf{Z}[\sqrt{2}]$ is a unique factorization domain.

Problem 5

a) Let $F = GF(2)(\alpha)$, where $\alpha^4 + \alpha + 1 = 0$. Determine $a, b, c, d \in GF(2)$ such that

$$\frac{1}{\alpha} = a + b\alpha + c\alpha^2 + d\alpha^3$$

b) Let $f(x)$ be an irreducible polynomial over $GF(p)$, where p is a prime. Show that $f(x)$ divides the polynomial $g(x) = x^{p^n} - x$ in $GF(p)[x]$ if and only if $deg(f(x))$ divides n.

Problem 6

Let L be a Galois extension of F such that $G(L|F)$ is abelian. Let $f(x) \in F[x]$ be the minimal polynomial of $\alpha \in L$. Show that all the roots of $f(x)$ lie in $F(\alpha)$.

Problem 7

Let $[E : F] < \infty$, where E is an extension of the field F. If $M_1(\subseteq E)$ and $M_2(\subseteq E)$ are two normal extensions of F show that M_1M_2 is a normal extension of F. (Here M_1M_2 denotes the subfield of E generated by M_1 and M_2.)