Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam: Sverre O. Smalø, phone 91750

Exam in MA3202 Galoisteori

English
Monday May 18 2009
Time: 09.00 - 13:00
Permitted aids: No printed or handwritten aids permitted.

Results: Friday June 5. 2009

Problem 1

Show that $\sqrt{-5}|(a+b\sqrt{-5})$ in $\mathbf{Z}[\sqrt{-5}]$ if and only if $5|a$, and use this to show that $\sqrt{-5}$ is a prime in $\mathbf{Z}\sqrt{-5}$.

Problem 2

Let F be a finite field such that $\text{char}(F) = 7$. Show that every element $a \in F$ ha a unique 7th root $\sqrt[7]{a}$, i.e. $b^7 = a$, where $b = \sqrt[7]{a}$.

Problem 3

a) Let $f(x) = x^3 + 27x + 6 \in \mathbf{Q}[x]$ and let E be the splitting field of $f(x)$ over $\mathbf{Q}$. Determine $G(E|\mathbf{Q})$.

(Hint: Investigate the real roots of $f(x)$.

b) Show that there exists one and only one field K such that $\mathbf{Q} \subsetneq K \subsetneq E$ and K is a normal extension of $\mathbf{Q}$.

Problem 4

a) Let $\begin{array}{c}K\\ |\\ F\end{array}$ be a Galois extension and let L be an intermediate field $\begin{array}{c}K\\ |\\ L\\ |\\ F\end{array}$. Show that $\begin{array}{c}K\\ |\\ L\end{array}$ is a Galois extension.

b) Give an example that shows that $\begin{array}{c}L\\ |\\ F\end{array}$ does not have to be a Galois extension.

Problem 5

Let $\begin{array}{c}K\\ |\\ F\end{array}$ be a Galois extension such that $G(K|F)$ is cyclic of order n and let σ be a generator for $G(K|F)$. Assume that F contains a primitive n'th root ω of unity. Let $\alpha \in K\backslash F$, and let $(\omega, \alpha) \neq 0$ be the Lagrange resolvent defined by

$$(\omega, \alpha) = \alpha + \omega\sigma(\alpha) + \cdots + \omega^{n-1}\sigma^{n-1}(\alpha)$$

a) Show that $a = \alpha + \sigma(\alpha) + \cdots + \sigma^{n-1}(\alpha)$ is an element in F.

b) Show that $K = F((\omega, \alpha))$.

c) Let $b = (\omega, \alpha)^n$. Show that $b \in F$ and that K is the splitting field of $x^n - b \in F[x]$ over F.

d) Give an argument why $x^n - b$ is an irreducible polynomial over F.