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Examination paper for **MA3202 Galois Theory**

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Problem 1 Consider $\mathbb{Q}$, and the real numbers $\sqrt[3]{2}$ and $\sqrt{1+\sqrt[3]{2}}$.

a) Find the Galois group of the splitting field of $X^3 - 2$ over $\mathbb{Q}$.

b) Find the minimal polynomial f of $\sqrt{1+\sqrt[3]{2}}$ over $\mathbb{Q}$.

c) Let E be the splitting field of the polynomial f in b). Show that the Galois group $G(E/\mathbb{Q})$ is not abelian, and that it contains a normal subgroup of index 6 contained in a normal subgroup of index 2.

Problem 2 Let q a power of a prime number and consider the number 900.

a) Prove that the Galois field $GF(q^{900})$ has three maximal subfields that contain $GF(q)$.

b) Prove that the polynomial $1/900(X^{900} - X^{450} - X^{300} - X^{180} + X^{150} + X^{90} + X^{60} - X^{30})$ evaluated in q gives the number of irreducible monic polynomials of degree 900 over $GF(q)$.

Problem 3 Let $\mathbb{Z}_2(T)$ be the field of rational functions in the variable T, and $K[S]$ be the ring of polynomials over the field K in the variable S.

a) Prove that the polynomial $Y^2 + Y + (X^3 + X^2 + 1)$ is an separable irreducible polynomial in $\mathbb{Z}_2(X)[Y]$ and determine the Galois group of the splitting field of this polynomial over $\mathbb{Z}_2(X)$

b) Prove that the polynomial $X^3 + X^2 + (Y^2 + Y + 1)$ is a separable irreducible polynomial in $\mathbb{Z}_2(Y)[X]$.

c) Let F be the splitting field over $\mathbb{Z}_2(Y)$ for the polynomial in b). Find the Galois group $G(F/\mathbb{Z}_2(Y))$.

Problem 4 Let p be a prime number.

a) Prove that there exist non-zero ring maps from $GF(p^n)$, the Galois field with p^n elements, to the ring of $n \times n$-matrices over $\mathbb{Z}_p$

b) How many such ring maps as in a) exist for $p = 2$, and $n = 3$ in a)?