NTNU
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Department of Mathematical Sciences

Examination paper for **MA3202 Galoisteori**

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Examination date: 27. mai 2017
Examination time (from–to): 09:00–13:00
Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Other information:
Grading: 16. juni 2017

Language: English
Number of pages: 2
Number of pages enclosed: 0

Checked by:

Date Signature

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Problem 1

a) Let $N = GF(7^{210})$ and $K = GF(7^7)$. Describe the Galois group $G(N \mid K)$ and list the fields E such that $K \subseteq E \subseteq N$.

b) Determine the number of monic irreducible polynomials of degree two over $F = GF(7)$.

Problem 2

a) Let $R = \mathbb{Z}\left[\sqrt{-7}\right] \left(= \left\{a + b\sqrt{-7} \mid a, b \in \mathbb{Z}\right\}\right)$. Show that 2 and $1 + \sqrt{-7}$ (as well as $1 - \sqrt{-7}$) are irreducible in R.

b) Show that R is not a unique factorization domain (UFD).

Problem 3

a) Let F, N and K be fields such that $N = F(\alpha)$, $K = F(\beta)$, and let $p(x) \in F[x]$, $q(x) \in F[x]$ be the minimal polynomials of α and β, respectively, over F. Assume degree$(p(x)) = 17$ and degree$(q(x)) = 16$. Show that $[F(\alpha, \beta) : F] = 17 \cdot 16 = 272$.

b) Let E be the splitting field of $f(x) = x^{17} - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$. Determine $[E : \mathbb{Q}]$. Explain your argument.

c) Let $G = G(E \mid \mathbb{Q})$ be the Galois group of E over $\mathbb{Q}$, where E is the field in **b)**. Show that there exists an intermediate field L, $\mathbb{Q} \subseteq L \subseteq E$, such that L corresponds by the Galois correspondence to a normal subgroup H of G of order 17. Explain your argument.

d) Show that there exists an intermediate field M, $\mathbb{Q} \subseteq M \subseteq E$, such that $[M : \mathbb{Q}] = 34$. [*Hint:* Use Sylov's Theorem.]

e) Show that G is non-abelian. [*Hint:* G abelian implies that all subgroups are normal.]

Problem 4 Let $K_1 = F(\alpha)$ and $K_2 = F(\beta)$ be two finite and normal extensions of F. Show that $K = F(\alpha, \beta)$ is a finite and normal extension of F.

Problem 5 Let N be a Galois extension of K such that $G(N \mid K)$ is abelian. Let $\alpha \in N$ and let $p(x) \in K[x]$ be the minimal polynomial of α over K. Show that all roots of $p(x)$ lie in $K(\alpha)$.