NTNU – Trondheim
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Examination paper for **MA3202 Galois theory**

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You must give reasons for all answers. You can answer in Norwegian if you prefer to.

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Problem 1 Let K be a field and R the subring of $K[x]$ consisting of the polynomials for which the coefficient of x is zero, i.e. the polynomials of the form

$$a_0 + a_2x^2 + \cdots + a_nx^n \quad (n = 0 \text{ or } n \geq 2).$$

a) What are the units in R? Show that x^2 and x^3 are irreducible elements in R.

b) Is R a UFD? (Hint: consider x^6.)

Problem 2 Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$.

a) Find the splitting field E of $f(x)$ over $\mathbb{Q}$. What is the degree $(E : \mathbb{Q})$?

b) Describe the Galois group $G(E \mid \mathbb{Q})$.

Problem 3 Let $E \supset F$ be a field extension of degree $(E : F) = n$.

a) Show that if n is a prime number, then there is no proper intermediate field between E and F (that is, no field K with $E \supset K \supset F$ and $E \neq K \neq F$). Deduce that if $\alpha \in E \setminus F$, then the minimal polynomial of α in $F[x]$ has degree n.

b) Let $E = F(\alpha, \beta)$, where α has minimal polynomial in $F[x]$ of degree d_1, and β has minimal polynomial in $F[x]$ of degree d_2. Show that if d_1 and d_2 are coprime (i.e. $\gcd(d_1, d_2) = 1$), then $(E : F) = d_1d_2$.

c) Give an example where α and β are as in (b), and such that $\alpha\beta$ has minimal polynomial in $F[x]$ of degree d_1 or d_2. (Hint: consider $F = \mathbb{Q}$ with $\alpha = 2^{1/3}$ and β a suitable root of unity.)

Problem 4 Let $f(x) = x^4 + 1 \in \mathbb{Q}[x]$, i.e. the cyclotomic polynomial $\Phi_8(x)$. Use the theory of cyclotomic polynomials to describe its splitting field E and the Galois group $G(E \mid \mathbb{Q})$.

Problem 5 Let $E = F(\alpha_1, \alpha_2)$ be a Galois extension of a field F, and let $K_1 = F(\alpha_1)$ and $K_2 = F(\alpha_2)$. Consider the subgroups $H_1 = G(E \mid K_1)$ and $H_2 = G(E \mid K_2)$ of the Galois group $G(E \mid F)$.

a) Show that $H_1 \cap H_2 = \{e\}$, that is, the intersection of H_1 with H_2 is the trivial subgroup of $G(E \mid F)$.

b) Suppose that each element $g_1 \in H_1$ maps K_2 to K_2, and that each element $g_2 \in H_2$ maps K_1 to K_1. Show that $g_1 g_2 = g_2 g_1$ for all $g_1 \in H_1, g_2 \in H_2$.

Problem 6 Let $E \supset F$ be a Galois extension of degree $(E : F)$.

a) Is it possible that $(E : F) = 4$ and that there are precisely two proper intermediate fields between E and F?

b) Suppose that $(E : F) = 6$ and that E is the splitting field of a polynomial of degree 3 (and a Galois extension of F). How many proper intermediate fields are there between E and F?.

Problem 7 Let $f(x) = x^5 - x - 1 \in \mathbb{Z}_5[x]$ and $E = \mathbb{Z}_5(\beta)$, where β is a root of $f(x)$.

a) Show that $\beta+1, \beta+2, \beta+3, \beta+4$ are also roots of $f(x)$. Deduce that $\beta \notin \mathbb{Z}_5$.

b) Define $\sigma \in G(E \mid \mathbb{Z}_5)$ by $\sigma(\beta) = \beta + 1$. Find the order of σ in $G(E \mid \mathbb{Z}_5)$, and describe the action of σ on the roots of $f(x)$.

c) Use the above and the FTGT to deduce that $f(x)$ is irreducible, and that $(E : \mathbb{Z}_5) = 5$.