Exam MA3202: Galois Theory
June 2, 2014

Problem 1

a) Write down the irreducible polynomials over $GF(2)(= \mathbf{Z}_2)$ of degrees two and three, respectively.

b) How many irreducible polynomials of degree four are there over $GF(2)$?

Problem 2

Let $\begin{array}{c} E \\ | \\ F \end{array}$ where $F = GF(5^3)$, $E = GF(5^{24})$. Describe the Galois group $G(E|F)$, and list the fields K such that $F \subseteq K \subseteq E$.

Problem 3

Let $f(x) \in F[x]$ be a non-zero polynomial over the field F with various properties as described below. Let $\alpha \in \overline{F}$, where $\overline{F}$ denotes the algebraic closure of F.

a) Let $f(\alpha) = 0$. Assume that whenever $g(\alpha) = 0$ for some non-zero $g(x) \in F[x]$, then degree $(f(x)) \leq$ degree $(g(x))$. Show that $f(x)$ is irreducible over F.

b) Show the converse of a), that is: Assume $f(x)$ is irreducible over F and $f(\alpha) = 0$. Let $g(\alpha) = 0$ for some non-zero $g(x) \in F[x]$. Show that degree $(f(x)) \leq$ degree $(g(x))$.

Problem 4

a) Let $\begin{array}{c} F(\theta) \\ | \\ F \end{array}$ and $\begin{array}{c} F(\gamma) \\ | \\ F \end{array}$ be two Galois extensions of the field F, where char $(F) = 0$. Show that $\begin{array}{c} F(\theta, \gamma) \\ | \\ F \end{array}$ is a Galois extension of F.

b) Assume $G(F(\theta)|F)$ and $G(F(\gamma)|F)$ are both abelian groups. Show that $G(F(\theta, \gamma)|F)$ is an abelian group.

Problem 5

a) Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbf{R}_+$. Find the minimal polynomial of α over $\mathbf{Q}$.

b) Show that $\mathbf{Q}(\alpha)$ is a normal extension of $\mathbf{Q}$.

(Hint: Consider $\alpha\sqrt{2-\sqrt{2}}$.)

c) Determine $G(\mathbf{Q}(\alpha)|\mathbf{Q})$ and all the intermediate fields K, where $\mathbf{Q} \underset{\neq}{\subset} K \underset{\neq}{\subset} \mathbf{Q}(\alpha)$.

(Hint: Consider $\sigma \in G(\mathbf{Q}(\alpha)|\mathbf{Q})$ such that $\sigma(\alpha) = \sqrt{2-\sqrt{2}}$.)

Problem 6

a) Let $R = \mathbf{Z}[2i] = \{a + 2bi | a, b \in \mathbf{Z}\}$. So R is a subintegral domain of the Gaussian integers $\mathbf{Z}[i] = \{a+bi | a, b \in \mathbf{Z}\}$. Show that 2 and $2i$ are irreducibles in R.

b) Show that R is not a UFD.

Problem 7

Show that $\sqrt{2} + \sqrt[3]{3} \notin \mathbf{Q}$.

(Hint: Consider an appropriate field extension of $\mathbf{Q}$.)