

Contact during the exam: Christian Skau, phone 91755



Exam in MA3202 Galoisteori

English

Saturday 19. may 2012

Time: 09.00 - 13:00

Permitted aids: No printed or handwritten aids permitted

Results: 15. june 2012

Problem 1

- Show that $2 + i$ is irreducible in $\mathbb{Z}[i] = \{m + ni | m, n \in \mathbb{Z}\}$, and that 5 is reducible in $\mathbb{Z}[i]$.
- Which of the numbers 11, 13 and 19 are irreducible in $\mathbb{Z}[i]$? Give reasons.

Problem 2

Let $\omega \in \mathbf{C}$ be a primitive 13th root of unity, and let $E = \mathbf{Q}(\omega)$.

- Show that E is a normal extension of \mathbf{Q} , and determine the cyclic Galois group $G = G(E|\mathbf{Q})$.
- How many proper intermediate fields $K, \mathbf{Q} \subsetneq K \subsetneq E$, are there? Give reasons.

Problem 3

Let $f(x) = x^7 - 2 \in \mathbf{Q}[x]$, and let E be the splitting field of $f(x)$ over \mathbf{Q} .

- Determine the order of the Galois group $G = G(E|\mathbf{Q})$.
- Show that there exists a non-normal subgroup H of G .
- Show that there exists a normal subgroup N of G such that G/N is abelian.

Problem 4

Let $E = \mathbb{Q}(\sqrt{2} - 3\sqrt{3})$.

- a) Show that E is a normal extension of \mathbb{Q} , and determine $G = G(E|\mathbb{Q})$.
- b) List all the intermediate fields $K, \mathbb{Q} \subseteq K \subseteq E$.

Problem 5

We denote by $GF(p^n)$ the finite field with p^n elements, where p is a prime.

- a) Let $f(x)$ be an irreducible polynomial of degree 36 over $GF(5)$. By referring to relevant theorems show that $f(x)$ divides the polynomial

$$x^{5^{36}} - x$$

in $GF(5)[x]$, and that $f(x)$ has distinct roots.

- b) Let E be the splitting field of $f(x)$ over $GF(5)$. Exhibit by a diagram the inclusion of the intermediate fields $K, GF(5) \subseteq K \subseteq E$.

Contact during the exam: Christian Skau, phone 91755



Exam in MA3202 Galoisteori

English

Friday May 27. 2011

Time: 09.00 - 13:00

Permitted aids: No printed or handwritten aids permitted

Results: Friday June 17. 2011

Problem 1

Show that $\mathbf{Z}[\sqrt{-13}]$ is not a unique factorization domain (UFD) by showing that $r = 6 + \sqrt{-13}$ is a divisor of $49 = 7 \cdot 7$, but r is not a divisor of 7.

Problem 2

- Let $N = GF(5^{30})$ be a field extension of $F = GF(5^2)$. Determine the number of intermediate fields K , i.e. fields K such that $F \subseteq K \subseteq N$.
- Determine the number of monic irreducible polynomials of degree 2 over $F = GF(5^2)$.

Problem 3

Let $f(x) = \frac{x^8}{8!} + \frac{x^7}{7!} + \cdots + \frac{x}{1!} + 1 = \sum_{k=0}^8 \frac{x^k}{k!} \in \mathbf{Q}[x]$. Let E be the splitting field of $f(x)$ over \mathbf{Q} .

One can show that the Galois groups $G(E|\mathbf{Q})$ is isomorphic to A_8 , the *alternating* group on 8 symbols. Use this to show that $f(x)$ is irreducible over \mathbf{Q} .

Problem 4

Let E be the splitting field over \mathbf{Q} of $f(x) = x^{13} - 2$, and let $G = G(E|\mathbf{Q})$ be the Galois group.

- Determine the order $|G|$ of G .
- Show that G is non-abelian.

- c) Show that there exists an intermediate field $K, \mathbf{Q} \subseteq K \subseteq E$, such that K corresponds by the Galois correspondence to a normal subgroup H of G such that the order $|H|$ of H is 13.
- d) Show that there exists an intermediate field N such that $[N : \mathbf{Q}] = 39$.

Problem 5

Let \mathbf{Q}_2 denote the subset of \mathbf{Q} consisting of all rational numbers $\frac{a}{b}$ such that $a, b \in \mathbf{Z}$ and b an odd number.

- a) Show that \mathbf{Q}_2 is a sub-integral domain of \mathbf{Q} , and find the units of \mathbf{Q}_2
- b) Show that the irreducible elements in \mathbf{Q}_2 consist of 2 and elements associated to 2.
- c) Is \mathbf{Q}_2 a unique factorization domain? Give reasons.

Faglig kontakt under eksamen: Christian Skau, telefon 91755



Eksamensordning

Eksamensordning

Eksamensordning

Bokmål

Fredag 21. mai 2010

Tid: 09.00 - 13:00

Hjelpeemidler: Ingen

Sensur: Fredag 11. juni 2010

Oppgave 1

La ω være en primitiv n'te enhetsrot, og la F være en kropp slik at $p \nmid n$, der $p = \text{char}(F)$. Vis at $F(\omega)$ er rotkroppen over F til $x^n - 1 \in F[x]$, og at Galoisgruppen $G(F(\omega)|F)$ er abelsk.

Oppgave 2

La $f(x) = x^3 + 9x - 2 \in \mathbb{Q}[x]$, og la E være rotkroppen til $f(x)$ over \mathbb{Q} . Vis at det finnes nøyaktig tre mellomkropper K , $\mathbb{Q} \subsetneq K \subsetneq E$, slik at $\frac{K}{\mathbb{Q}}$ ikke er en normal utvidelse.

Oppgave 3

Ved veltemperert stemming av et piano trenger man å bestemme forholdet $\alpha = \sqrt[12]{2} \in \mathbf{R}_+$. Begrunn hvorfor α ikke kan konstrueres fra \mathbb{Q} med passer og linjal.

Oppgave 4

La F være en kropp av karakteristikk 0. La $f(x) \in F[x]$ være et polynom av grad n med n distinkte røtter $\alpha_1, \alpha_2, \dots, \alpha_n$. La

$$\beta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + n\alpha_n$$

Anta at man får $n!$ forskjellige verdier dersom man permuterer $1, \dots, n$, dvs. mengden

$$\{k_1\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n \mid k_1 k_2 \dots k_n \text{ er en permutasjon av } 12 \dots n\}$$

består av $n!$ distinkte tall.

Vis at $F(\beta)$ er rotkroppen til $f(x)$ over F . [Hint: Benytt hovedteoremet i Galoisteorien.]

Oppgave 5

- a) La $E = GF(3^{12})$ være en kroppsutvidelse av $F = GF(3^4)$. Hvor mange mellomkropper finnes det mellom F og E ?
- b) Bestem antall irreducible moniske polynomer av grad 2 over $GF(7)$.

Oppgave 6

La $\alpha = \sqrt{5 + \sqrt{5}} \in \mathbf{R}_+$, og la $E = \mathbf{Q}(\alpha)$.

- a) Vis at $\sqrt{5 - \sqrt{5}} \in E$.
(Hint: Betrakt $\frac{1}{\alpha}$.)
- b) Vis at E er rotkroppen til et polynom $f(x) \in \mathbf{Q}[x]$.
- c) Bestem Galoisgruppen $G = G(E|\mathbf{Q})$ (som abstrakt gruppe), ved først å bestemme ordenen til G .

Contact during the exam: Sverre O. Smalø, phone 91750



Exam in MA3202 Galoisteori

English

Monday May 18 2009

Time: 09.00 - 13:00

Permitted aids: No printed or handwritten aids permitted.

Results: Friday June 5. 2009

Problem 1

Show that $\sqrt{-5}|(a + b\sqrt{-5})$ in $\mathbf{Z}[\sqrt{-5}]$ if and only if $5|a$, and use this to show that $\sqrt{-5}$ is a prime in $\mathbf{Z}\sqrt{-5}$.

Problem 2

Let F be a finite field such that $\text{char}(F) = 7$. Show that every element $a \in F$ has a unique 7th root $\sqrt[7]{a}$, i.e. $b^7 = a$, where $b = \sqrt[7]{a}$.

Problem 3

- a) Let $f(x) = x^3 + 27x + 6 \in \mathbf{Q}[x]$ and let E be the splitting field of $f(x)$ over \mathbf{Q} . Determine $G(E|\mathbf{Q})$.

(Hint: Investigate the real roots of $f(x)$.)

- b) Show that there exists one and only one field K such that $\mathbf{Q} \subsetneq K \subsetneq E$ and K is a normal extension of \mathbf{Q} .

Problem 4

- a) Let $\begin{array}{c} K \\ | \\ F \end{array}$ be a Galois extension and let L be an intermediate field $\begin{array}{c} L \\ | \\ K \end{array}$. Show that $\begin{array}{c} K \\ | \\ L \\ | \\ F \end{array}$ is a Galois extension.

- b) Give an example that shows that $\begin{array}{c} L \\ | \\ F \end{array}$ does not have to be a Galois extension.

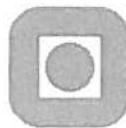
Problem 5

Let $\begin{array}{c} K \\ | \\ F \end{array}$ be a Galois extension such that $G(K|F)$ is cyclic of order n and let σ be a generator for $G(K|F)$. Assume that F contains a primitive n 'th root ω of unity. Let $\alpha \in K \setminus F$, and let $(\omega, \alpha) \neq 0$ be the Lagrange resolvent defined by

$$(\omega, \alpha) = \alpha + \omega\sigma(\alpha) + \cdots + \omega^{n-1}\sigma^{n-1}(\alpha)$$

- a) Show that $a = \alpha + \sigma(\alpha) + \cdots + \sigma^{n-1}(\alpha)$ is an element in F .
- b) Show that $K = F((\omega, \alpha))$.
- c) Let $b = (\omega, \alpha)^n$. Show that $b \in F$ and that K is the splitting field of $x^n - b \in F[x]$ over F .
- d) Give an argument why $x^n - b$ is an irreducible polynomial over F .

Faglig kontakt under eksamen: Professor Christian Skau
(telefon: 91755)



Eksamensordning

Eksamnen i MA3202 Galoisteori

Torsdag 1. juni 2006
Tid: 09.00 - 13:00
Hjelpeemidler: Ingen
Bokmål

Sensur: 22. juni 2006

Oppgave 1

La $f(x) \in F[x]$ være et irreduksibelt polynom over kroppen F , der F har karakteristikk 0. La E være rotkroppen til $f(x)$ over F , og anta at Galoisgruppen $G = G(E|F)$ er abelsk.

Vis at hver rot α til $f(x)$ er et primitivt element, dvs. $E = F(\alpha)$.

Oppgave 2

Vis at den diofantiske ligningen

$$y^2 + 2 = z^4$$

ikke har noen løsninger i \mathbb{Z} .

[Hint: $\mathbb{Z}[\sqrt{-2}]$ er et entydig faktoriseringssområde.]

Oppgave 3

La F være en kropp av karakteristikk p , der p er et primtall. La $f(x) \in F[x]$ være et irreduksibelt polynom med multiple røtter. Vis at det finnes $s \in \{1, 2, 3, \dots\}$ og et irreduksibelt og *separabelt* polynom $g(x) \in F[x]$, slik at $f(x) = g(x^{p^s})$.

Oppgave 4

- a) La K være en Galoisk utvidelse av F . La $g(x) \in K[x]$ være irreduksibel over K , og la $\sigma \in G(K|F)$.

Vis at $\sigma(g(x)) \in K[x]$ er irreduksibel over K .

- b) La $f(x) \in F[x]$ være et (monisk) irreduksibelt polynom over F av primtallsgrad p .

Vis at dersom $f(x)$ er redusibel i $K[x]$, så vil alle røttene til $f(x)$ ligge i K .

[F og K er som i a).]

Oppgave 5

La $f(x) = x^3 - 21x + 6 \in \mathbb{Q}[x]$, og la E være rotkroppen til $f(x)$ over \mathbb{Q} . Man kan vise at $E \subset \mathbb{R}$. Vis at E ikke er et radikaltårn over \mathbb{Q} .



Contact during til exam:
Christian Skau, Telefon: 91755

MA3202 Commutativ algebra and Galois theory
English
Friday May 14, 2004
Kl. 9-13
Permitted aids: None
Grades to be announced: Monday May 24, 2004

Problem 1

- Prove that if D is a domain that is not a field, then $D[x]$ is not a Euclidean domain.
- Show that 3 is irreducible, but not prime, in the integral domain $\mathbb{Z}[\sqrt{-5}]$.

Problem 2

- Determine the Galois group of $x^3 - 7 \in \mathbb{Q}[x]$ over \mathbb{Q} .
- Let E denote the splitting field of $x^3 - 7$ over \mathbb{Q} . How many intermediate fields $F(\mathbb{Q} \subset F \subset E)$, such that $[F : \mathbb{Q}] = 2$, are there? Give reasons.

Problem 3

Let p be a prime. Let E be the splitting field of $x^p - 1 \in \mathbb{Q}[x]$ over \mathbb{Q} .

- Prove that $G(E/\mathbb{Q})$ is abelian of order $p - 1$.
- Let $\omega = e^{\frac{2\pi i}{31}}$. Prove that there exists a subfield F of \mathbb{C} such that $[F(\omega) : F] = 5$.

Problem 4

- a) Let F be a field of characteristic p , where $0 < p \neq 3$. Let α be a root of $f(x) = x^p - x + 3 \in F[x]$ that lies in F . Show that $f(x)$ has p distinct roots in F .
[HINT: Show that $\alpha + 1$ is a root.]
- b) Without actually computing, find the number of monic irreducible polynomials of degree 2 over the field $\mathbb{Z}_7 = GF(7)$.

Problem 5

Prove that $\sqrt{2} + \sqrt[3]{3}$ is irrational.

[HINT: Consider $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{3})$.]