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MA3202 Commutativ algebra and Galois theory

English
Friday May 14, 2004
Kl. 9-13
Permitted aids: None
Grades to be announced: Monday May 24, 2004

Problem 1

a) Prove that if D is a domain that is not a field, then $D[x]$ is not a Euclidean domain.

b) Show that 3 is irreducible, but not prime, in the integral domain $\mathbb{Z}[\sqrt{-5}]$.

Problem 2

a) Determine the Galois group of $x^3 - 7 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.

b) Let E denote the splitting field of $x^3 - 7$ over $\mathbb{Q}$. How many intermediate fields $F(\mathbb{Q} \subset F \subset E)$, such that $[F : \mathbb{Q}] = 2$, are there? Give reasons.

Problem 3

Let p be a prime. Let E be the splitting field of $x^p - 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.

a) Prove that $G(E/\mathbb{Q})$ is abelian of order $p - 1$.

b) Let $\omega = e^{\frac{2\pi i}{31}}$. Prove that there exists a subfield F of $\mathbb{C}$ such that $[F(\omega) : F] = 5$.

Problem 4

a) Let F be a field of characteristic p, where $0 < p \neq 3$. Let α be a root of $f(x) = x^p - x + 3 \in F[x]$ that lies in F. Show that $f(x)$ has p distinct roots in F.

[HINT: Show that $\alpha + 1$ is a root.]

b) Without actually computing, find the number of monic irreducible polynomials of degree 2 over the field $\mathbb{Z}_7 = GF(7)$.

Problem 5

Prove that $\sqrt{2} + \sqrt[3]{3}$ is irrational.

[HINT: Consider $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{3})$.]