MA8201 Exam 15.12.2025

Problem 1

(a) Doing elementary row/column op's to obtain SNF:

$$\begin{bmatrix}-5&1&12\\6&0&-12\\-4&4&14\end{bmatrix} \sim \begin{bmatrix}1&-5&12\\0&6&-12\\4&-4&14\end{bmatrix}\begin{matrix}-4\\ \\ \\ \end{matrix} \sim \begin{array}{c}\begin{bmatrix}1&-5&12\\0&6&-12\\0&16&-34\end{bmatrix}\\ 5 \\ -12\end{array}$$

$$\begin{array}{c}\begin{bmatrix}1&0&0\\0&6&-12\\0&16&-34\end{bmatrix}\\ 2\end{array} \sim \begin{array}{c}\begin{bmatrix}1&0&0\\0&6&0\\0&16&-2\end{bmatrix}\\ 8\end{array} \sim \begin{bmatrix}1&0&0\\0&6&0\\0&0&-2\end{bmatrix}\cdot -1$$

$$\begin{bmatrix}1&0&0\\0&6&0\\0&0&2\end{bmatrix} \sim \underline{\underline{\begin{bmatrix}1&0&0\\0&2&0\\0&0&6\end{bmatrix}}}$$

where we have $1 \mid 2 \mid 6$, such that this is the Smith normal form of A.

(b) By (a) we know there exist two invertible 3×3-matrices P and Q such that

$$PAQ = \begin{bmatrix}1&0&0\\0&2&0\\0&0&6\end{bmatrix} = D$$

From the theory we have that

$$\mathbb{Z}^3/\operatorname{Im} f_A \simeq \mathbb{Z}^3/\operatorname{Im} f_D$$

$$\simeq \mathbb{Z}^3 \Big/ \operatorname{Im}\begin{bmatrix}1&0&0\\0&2&0\\0&0&6\end{bmatrix} = \mathbb{Z}^3 / (\mathbb{Z}\oplus 2\mathbb{Z}\oplus 6\mathbb{Z})$$
$$\simeq \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$$
$$\simeq \underline{\underline{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.}}$$

Problem 2

(a) By the theory it is enough to show that

(i) $r, s \in R \Rightarrow \begin{cases} r-s \in R, \\ rs \in R. \end{cases}$

(ii) $1_{M_3(k)} = I_3 \in R.$

Let
$$r = \begin{bmatrix}a&b&c\\0&d&e\\0&f&g\end{bmatrix} \text{ and } s = \begin{bmatrix}a'&b'&c'\\0&d'&e'\\0&f'&g'\end{bmatrix}.$$

Clearly,
$$r-s = \begin{bmatrix}a-a'&b-b'&c-c'\\0&d-d'&e-e'\\0&f-f'&g-g'\end{bmatrix} \in R \qquad (*)$$

and
$$r\cdot s = \begin{bmatrix}a&b&c\\0&d&e\\0&f&g\end{bmatrix}\begin{bmatrix}a'&b'&c'\\0&d'&e'\\0&f'&g'\end{bmatrix} = \begin{bmatrix}*&*&*\\0&*&*\\0&*&*\end{bmatrix} \in R.$$

Only need to check if these two entries are zero.

By definition of R
$$1_{M_3(k)} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \in R.$$

Hence R is a subring of $M_3(k)$.

(a) & (b): Define $\varphi : R \longrightarrow k \oplus \begin{bmatrix} k & k \\ k & k \end{bmatrix}$ by letting

$$\varphi\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix}\right) = \left(a, \begin{bmatrix} d & e \\ f & g \end{bmatrix}\right) \quad (**)$$

By (*) above, we see that $\varphi(r+s) = \varphi(r) + \varphi(s)$, (substituting - with +), and

$$\varphi(r \cdot s) = \varphi\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix}\begin{bmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & f' & g' \end{bmatrix}\right)$$

$A = \begin{bmatrix} d & e \\ f & g \end{bmatrix}$, $A' = \begin{bmatrix} d' & e' \\ f' & g' \end{bmatrix}$

$$= \varphi\left(\begin{bmatrix} aa' & * & * \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$[A \cdot A']$}} \\ 0 & & \end{bmatrix}\right) = (aa', A \cdot A')$$

$$= (a, A)(a', A') = \varphi(r)\varphi(s)$$

Furthermore,

$$\varphi\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \left(1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1_{k \oplus \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}}$$

Hence, φ is a homomorphism of rings. We see from the definition of φ that φ is onto, since (**) is an arbitrary element in $k \oplus \begin{bmatrix} k & k \\ k & k \end{bmatrix}$.

Also

$$\operatorname{Ker} \varphi = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix} \in R \;\middle|\; \varphi\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix}\right) = \left(a, \begin{bmatrix} d & e \\ f & g \end{bmatrix}\right) = \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \right\}$$

$$= \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R \;\middle|\; b, c \in k \right\} = I$$

By the theory $\operatorname{Ker} \varphi = I$ is an ideal in R, and

$$R/I = R/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi = k \oplus \begin{bmatrix} k & k \\ k & k \end{bmatrix}.$$

The ring R/I is a finite direct sum of full matrix rings over division rings, hence by Wedderburn-Artin theorem R/I is a semisimple ring.

(c) We have that $e^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = e,$
and e is an idempotent.
Let $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then $Re_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $Re_2 = \begin{bmatrix} 0 & k & 0 \\ 0 & k & 0 \\ 0 & k & 0 \end{bmatrix}$ and $Re_3 = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 0 & 0 & k \end{bmatrix}$,
and it follows that

$$R = Re_1 + Re_2 + Re_3. \qquad (\Delta)$$

Since
$$Re_1 \cap (Re_2 + Re_3) = (0),$$
$$Re_2 \cap (Re_1 + Re_3) = (0), \text{ and}$$
$$Re_3 \cap (Re_1 + Re_2) = (0),$$
by the theory (Δ) is a direct sum and
$$R = Re_1 \oplus Re_2 \oplus Re_3.$$

Problem 3

(a) We need to show that

(i) $r, r' \in \operatorname{Ann}_R(M) \Rightarrow r - r' \in \operatorname{Ann}_R(M)$.

(ii) $r \in \operatorname{Ann}_R(M)$, $s \in R \Rightarrow rs, sr \in \operatorname{Ann}_R(M)$.

(i) Let $r, r' \in \operatorname{Ann}_R(M)$. For $m \in M$, then
$$(r - r')m = \underset{\substack{\shortparallel \\ 0}}{rm} - \underset{\substack{\shortparallel \\ 0}}{r'm} = 0$$
, since $r, r' \in \operatorname{Ann}_R(M)$

Hence, (i) holds. Let $r \in Ann_R(M)$ and $s \in R$, then for $m \in M$, $(sr)\cdot m = s(\underset{\overset{\shortparallel}{0}\text{, since } r\in Ann_R(M)}{r\cdot m}) = s\cdot 0 = 0$

and $(r\cdot s)m = \underbrace{r(\underbrace{sm}_{\in M})}_{=0} = 0$

Hence (ii) holds, and $Ann_R(M)$ is an ideal in R.

(b) <u>f onto</u>: Then for any $n \in N$, there exists $m \in M$ such that $f(m) = n$. Let $r \in Ann_R(M)$. Then

$$r\cdot n = rf(m) = f(\underbrace{rm}_{=0\text{, since}}) = 0$$

since f is an R-homomorphism.

Hence $r\cdot n = 0$ for all $n \in N$, and $Ann_R(M) \subseteq Ann_R(N)$.

<u>f 1-1</u>: Then $Ker(f) = (0)$. Let $m \in M$ and $r \in Ann_R(N)$. Then $f(rm) = rf(m) = 0$, since $r \in Ann_R(N)$ and $f(m) \in N$. Hence, $rm \in Ker(f) = (0)$ and $r \in Ann_R(M)$. It follows that $Ann_R(N) \subseteq Ann_R(M)$.

It follows from the above that if f is an isomorphism, then $Ann_R(M) = Ann_R(N)$. The contrapositive statements is true:

$$Ann_R(M) \neq Ann_R(N) \Rightarrow f \text{ is not an isomorphism.}$$

<u>Problem 4</u>

(a) By definition

$$M_{p(x)^n} = \{m \in M \mid p(x)^n\cdot m = 0\} = \{m \in M \mid p(h)^n(m) = 0\}$$

definition of the $k[x]$-action

$$= \underline{\underline{Ker(p(h)^n)}}.$$

Since $\mathrm{End}_\Lambda(M)$ is a ring, any polynomial in an endomorphism g of M (interpreting $g^0 = 1_M$) is again an endomorphism of M in $\mathrm{End}_\Lambda(M)$. Since the kernel of any Λ-endomorphism of M is a Λ-submodule of M, it follows that $\mathrm{Ker}(p(h)^n)$ is a Λ-submodule of M.

(b)(i) Let $\varphi : M \longrightarrow X$ be an isomorphism of left $k[x]$-modules. Claim: $\varphi|_{M_a} : M_a \longrightarrow X_a$ is an isomorphism. For $m \in M_a$, we have $a\varphi(m) = \varphi(am) = \varphi(0) = 0$, since $m \in M_a$. Hence, $\varphi(m) \in X_a$. Let $x \in X_a$. Then $\exists\, m \in M$ such that $\varphi(m) = x$. Since $0 = ax = a\varphi(m) = \varphi(am)$ and φ is 1-1, we infer that $m \in M_a$. It follows that $\varphi|_{M_a}$ is onto and a $k[x]$-homomorphism and 1-1, since φ is. Hence, $\varphi|_{M_a} : M_a \to X_a$ is an isomorphism of $k[x]$-modules.

(ii) Given that

$$M \simeq k[x]/(p_1(x)^{n_1}) \oplus \cdots \oplus k[x]/(p_t(x)^{n_t})$$

it follows from (I) in the problem text, (b)(i) and the formula for $X_{p(x)^n}$ in (b)(ii) that

$$M_{p_i(x)^n} \simeq \bigoplus_{\{i \mid p_i(x) = p_i(x)\}} k[x]/(p_i(x)^{n_i}),$$

since $n \geq n_i$ for all i.

(iii) By using (ii) it follows that

$$M = \bigoplus_{i=1}^{s} M_{p_i(x)^n} \simeq \bigoplus_{i=1}^{s} \Big(\bigoplus_{\{j \mid p_j(x) = p_i(x)\}} k[x]/(p_j(x)^{n_j}) \Big),$$

as left $k[x]$-modules.

(c) Let $h \in \mathrm{End}_\Lambda(M)$ be such that the characteristic polynomial $c_h(x)$ of h has at least two different monic irreducible factors. Recall from the theory that in (2) in the problem

$$\prod_{i=1}^{t} p_i(x)^{n_i} = \pm c_h(x).$$

Hence in the decomposition $M = \bigoplus_{i=1}^{s} M_{p_i(x)^n}$ there are at least two non-zero terms. By (a) each $M_{p_i(x)^n}$ is a Λ-submodule of M, and we infer that $M = M_1 \oplus M_2$ with $M_i \neq (0)$.

If $M = M_1 \oplus M_2$ for two non-zero submodules M_1 and M_2, define $h : M \longrightarrow M$ by letting

$$h((m_1, m_2)) = (m_1, 0).$$

Then $h \in \mathrm{End}_\Lambda(M)$ and as a linear transformation h can be represented as

$$h = \left[\begin{array}{c|c} I_{\dim_k M_1} & 0 \\ \hline 0 & 0_{\dim_k M_2} \end{array}\right]$$

Then $c_h(x) = (x-1)^{\dim_k M_1} x^{\dim_k M_2}$. Since both M_1 and M_2 are non-zero, $c_h(x)$ has two different monic irreducible factors. This completes the proof.