

Problem 1 Let A be the 3×3 -matrix over the integers \mathbb{Z} given by

$$A = \begin{bmatrix} -5 & 1 & 12 \\ 6 & 0 & -12 \\ -4 & 4 & 14 \end{bmatrix}$$

- a) Find the Smith normal form of A over \mathbb{Z} .
- b) View A as a homomorphism $f_A: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ given by

$$f_A \left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) = A \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

(You can assume known that f_A is a homomorphism of \mathbb{Z} -modules). Find the \mathbb{Z} -module (or abelian group) $\mathbb{Z}^3 / \text{Im}(f_A)$ as a direct sum of \mathbb{Z} -modules of the form $\mathbb{Z}/(n)$ for appropriate $n \geq 0$, where $\text{Im}(f_A)$ denotes the image of the homomorphism f_A .

Problem 2 Let k be a field, and

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix} \mid a, b, c, d, e, f, g \in k \right\}.$$

- a) Show that R is a subring of $M_3(k) = \begin{bmatrix} k & k & k \\ k & k & k \\ k & k & k \end{bmatrix}$, and that

$$I = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid b, c \in k \right\}$$

is an ideal in R .

- b) Show that R/I is isomorphic to the ring

$$k \oplus \begin{bmatrix} k & k \\ k & k \end{bmatrix}.$$

Is R/I a semisimple ring?

- c) Show that $e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is an idempotent in R . Find three idempotents $\{e_1, e_2, e_3\}$ in R such that

$$R = Re_1 \oplus Re_2 \oplus Re_3$$

as a left R -module.

Problem 3 Let R be a ring, and let M and N be two left R -modules.

- a) Let

$$\text{Ann}_R(M) = \{r \in R \mid rm = 0, \forall m \in M\}.$$

Show that $\text{Ann}_R(M)$ is a two-sided ideal in R .

- b) Let $f: M \rightarrow N$ be an R -homomorphism of the left R -modules M and N . Show that

- (i) if f is onto, then $\text{Ann}_R(M) \subseteq \text{Ann}_R(N)$,
- (ii) if f is 1-1, then $\text{Ann}_R(M) \supseteq \text{Ann}_R(N)$.

Show that if $\text{Ann}_R(M) \neq \text{Ann}_R(N)$, then M and N are not isomorphic as left R -modules.

Problem 4 Let $h: M \rightarrow M$ be a Λ -homomorphism (endomorphism) of a left Λ -module M for a finite dimensional algebra Λ over a field k . Then h is in particular a linear transformation of the vector space M , and M becomes a (left) $k[x]$ -module by letting $f(x)$ in $k[x]$ act on M by defining

$$f(x) \cdot m = f(h)(m)$$

for all $m \in M$. As defined in the course, for any $k[x]$ -module X and any element a in $k[x]$ let

$$X_a = \{x \in X \mid ax = 0\}.$$

Furthermore, it was shown that $X_a \subseteq X$ is a $k[x]$ -submodule of X , and that

$$(X \oplus X')_a = X_a \oplus X'_a \tag{1}$$

for two left $k[x]$ -modules X and X' .

a) For any non-zero polynomial $g(x)$ in $k[x]$, show that

$$M_{g(x)} = \text{Ker}(g(h))$$

when viewing M as a $k[x]$ -module. Using that the set of endomorphisms of M ,

$$\text{End}_\Lambda(M) = \{f: M \rightarrow M \mid f \text{ is a } \Lambda\text{-homomorphism of } M\},$$

is a ring, explain why $\text{Ker}(g(h))$ also is a Λ -submodule of M .

b) (i) Let $\varphi: M \rightarrow X$ be an isomorphism as left $k[x]$ -modules. Show that

$$M_a \simeq X_a$$

as $k[x]$ -modules for any non-zero element a in $k[x]$.

(ii) For two polynomials $f(x)$ and $g(x)$ in $k[x]$, denote by $f(x) \nmid g(x)$ the fact that $f(x)$ does not divide $g(x)$.

Let $p(x)$ and $q(x)$ be two monic irreducible polynomials in $k[x]$. For a positive integer m consider the factor module $X = k[x]/(q(x)^m)$ of $k[x]$ as a module over $k[x]$. For a positive integer n with $n \geq m$ we proved in the course that

$$X_{p(x)^n} = \begin{cases} (0), & \text{if } q(x) \nmid p(x), \text{ or equivalently } p(x) \neq q(x), \\ X, & \text{if } q(x) \mid p(x), \text{ or equivalently } p(x) = q(x). \end{cases}$$

From the theory, we know that as a $k[x]$ -module

$$M \simeq k[x]/(p_1(x)^{n_1}) \oplus k[x]/(p_2(x)^{n_2}) \oplus \cdots \oplus k[x]/(p_t(x)^{n_t}) \quad (2)$$

for some monic irreducible polynomials $p_i(x)$ and $n_i \geq 1$ for $i = 1, 2, \dots, t$.

For $n = \max\{n_i\}_{i=1}^t$ prove that

$$M_{p_1(x)^n} \simeq \bigoplus_{\{i \mid p_i(x) = p_1(x)\}} k[x]/(p_i(x)^{n_i})$$

as a $k[x]$ -module.

(iii) Renumber the monic irreducible polynomials $\{p_i(x)\}_{i=1}^t$ such that $\{p_i(x)\}_{i=1}^s$ are all the different monic polynomials among $\{p_i(x)\}_{i=1}^t$ for some positive integer $s \leq t$. Prove for $n = \max\{n_i\}_{i=1}^t$ that

$$M = \bigoplus_{i=1}^s M_{p_i(x)^n}$$

as a left $k[x]$ -module.

c) Prove that, as a Λ -module,

$$M = M_1 \oplus M_2$$

for two non-zero submodules M_1 and M_2 of M if and only if there is an h in $\text{End}_\Lambda(M)$ such that the characteristic polynomial $c_h(x)$ of h as a linear transformation of the vector space M has at least two different monic irreducible polynomials $p_1(x)$ and $p_2(x)$ as factors, that is, $p_1(x)p_2(x) \mid c_h(x)$.