Rings and modules - Problem set 6

To be solved on Tuesday 21.11

Problem 1. (After Chapter 14.2.) Let R be a domain (i.e. a commutative unital integral domain) and M an R-module. Show that the set

$$\operatorname{Tor} M = \{m \in M \mid \text{there exists nonzero } r \in R \text{ with } rm = 0\}$$

is a submodule of M.

Problem 2. (After Chapter 21.4.) Let K be a field and $A \in M_{n\times n}(K)$ be an $n \times n$ matrix with coefficients in K. Show that there exists a unique monic polynomial $m_A(X) \in K[X]$ which generates the ideal $I_A = \{p(X) \in K[X] \mid p(A) = 0\}$.

Problem 3. (After Chapter 20.3.) (Exam August 2013, Problem 1.) Find the Smith normal form over the integers $\mathbb{Z}$ for the matrix $\begin{pmatrix} 4 & 2 & 4 \\ 12 & -6 & 6 \\ -16 & 10 & -4 \end{pmatrix}$.

Problem 4. (After Chapter 21.4.) (Exercise 21.4.1 in the book.) Find rational canonical forms of the following matrices over $\mathbb{Q}$:

(a) $\begin{pmatrix} 1 & 5 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 2 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 8 & 3 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Problem 5. (After Chapter 21.5) (Exercise 21.5.1 in the book.) Find the Jordan canonical form of the following matrices:

(a) $\begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}$.

(b) $\begin{pmatrix} 5 & \frac{1}{2} & -2 & 4 \\ 0 & 5 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Problem 6. (After Chapter 21.4.) (Exam August 2013, Problem 2.) Let A be a 4×4 matrix over $\mathbb{R}$ with minimal polynomial $m_A(X) = (X - 3)^2$. What are the possible forms that the rational canonical form of A can have?

Problem 7. (After Chapter 21.4.) (Exam December 2013, Problem 1.)

(a) Find the Smith normal form of the matrix $\begin{pmatrix} 2-X & 1 & 2 \\ 0 & 1-X & 2 \\ 1 & 0 & 1-X \end{pmatrix}$ over $\mathbb{Z}_3[X]$.

(b) Find the rational canonical form of the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}_3$.

(c) Let $M_{3\times 3}(\mathbb{Z}_3)$ be the 3×3 matrix ring over $\mathbb{Z}_3$ and define $\Phi_A : \mathbb{Z}_3[X] \to M_{3\times 3}(\mathbb{Z}_3)$ by letting $\Phi_A(P(X)) = P(A)$ for each polynomial $P(X)$ in $\mathbb{Z}_3[X]$. The image of Φ_A is then the subring of $M_{3\times 3}(\mathbb{Z}_3)$ generated by the matrix A. Prove that this subring is a field.

Problem 8. (After Chapter 21.2.) Let R be a PID and $A, B \in M_{n\times m}(R)$ be $n \times m$ matrices with coefficients in R. Let $C(A)$ respectively $C(B)$ be the column module of A respectively B, that is the submodule of R^n generated by the columns of A respectively B.

(a) Show that

$$C(A) = \left\{ A \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \middle| \; r_1, \dots, r_m \in R \right\}.$$

(b) Let $Q \in M_{m\times m}(R)$ be an invertible $m \times m$ matrix. Show that if $A = BQ$, then $C(A) = C(B)$.

(c) Let $Q \in M_{n\times n}(R)$ be an invertible $n \times n$-matrix. Let $T_P : R^n \to R^n$ be the R-module homomorphism given by $T_P \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = P \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$. Show that if $A = PB$, then T_P gives an isomorphism from $C(B)$ to $C(A)$.

(d) Show that if A is equivalent to B then $\frac{R^m}{C(A)}$ is isomorphic to $\frac{R^m}{C(B)}$.

(e) Use the Smith normal form to show that the converse of part (d) also holds, that is if $\frac{R^m}{C(A)}$ is isomorphic to $\frac{R^m}{C(B)}$, then A is equivalent to B.

Problem 9. (After Chapter 21.4.) Let K be a field and let $A \in M_{n\times n}(K)$ be an $n \times n$ matrix with coefficients in K. Consider the $K[X]$-module M_A where $M_A = K^n$ and for $p(X) \in K[X]$ and $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in K^n$ the multiplication is given by

$$p(X) \cdot_A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := p(A) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Define a map $\phi : K[X]^n \to M_A$ by

$$\phi((p_1(X), \dots, p_n(X))) = p_1(A) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + p_n(A) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that ϕ is a homomorphism of $K[X]$-modules.

(b) Show that $\operatorname{Im} \phi = M_A$.

(c) Show that $\ker \phi = C(A - XI_n)$, where $C(A - XI_n)$ is the column module of the matrix $A - XI_n \in M_{n\times n}(K[X])$. *(Hint: Show first that $C(A - XI_n) \subseteq \ker \phi$. Then for $(p_1(X), \dots, p_n(X)) \in \ker \phi$, show that $(p_1(X), \dots, p_n(X)) \in C(A - XI_n)$ by setting $\tilde{p}_j(X) = \frac{p_j(X) - b_{j0}}{X}$ and using induction on $d = \max\{\deg(p_1(X), \dots, p_n(X))\}$. This may be a bit challenging.)*

(d) Conclude that $\frac{K[X]^n}{C(A - XI_n)} \cong M_A$.

Problem 10. (After Chapter 14.3.) Let R be a PID and let $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subsetneq R$ be ideals in R. Assume that N is an R-module such that

$$N \cong R/I_1 \times \dots \times R/I_n.$$

Show that the following hold.

(a) There exists a generating set of N with n elements and every generating set of N has at least n elements. *(Hint: to show that every generating set of N has at least n elements, let M be a maximal ideal in R containing I_n and show that every generating set of N gives a generating set of $(R/M)^n$. Then, use the fact that R/M is a field to conclude that any generating set of $(R/M)^n$ has at least n elements.)*

(b) Let $r \in R$ and let $\phi_r : R \to R$ be the map $\phi_r(x) = rx$. Show that

$$rN \cong R/\phi_r^{-1}(I_1) \times \cdots \times R/\phi_r^{-1}(I_n).$$

(c) For $1 \leq i \leq n$ we have that

$$I_i = \{r \in R \mid rN \text{ can be generated by fewer than } i \text{ elements}\}.$$

(Hint: apply part (a) to part (b).)

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 11. (After Chapter 21.4.) Let F be a field and assume that F is a subfield of another field K. Let $A, B \in M_{n\times n}(F)$ be two $n \times n$ matrices with coefficients in F. Show that A and B are similar over $M_{n\times n}(F)$ if and only if they are similar over $M_{n\times n}(K)$.

Problem 12. (After Chapter 21.5.) For a matrix $M \in M_{n\times n}(\mathbb{R})$, we define

$$\exp(M) = e^M := I_n + M + \tfrac{1}{2!}M^2 + \tfrac{1}{3!}M^3 + \cdots.$$

(It can be shown that this is well-defined.) Let $y_1(t), \ldots, y_n(t)$ be functions from $\mathbb{R}$ to $\mathbb{R}$ and let $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$. Consider the initial value problem

$$\begin{cases} \mathbf{y}'(t) & = A\mathbf{y}(t), \\ \mathbf{y}(0) & = y_0, \end{cases}$$

where $A \in M_{n\times n}(\mathbb{R})$, $y_0 \in \mathbb{R}^n$ and $\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$. It can be shown that this problem has a unique solution given by $\mathbf{y}(t) = e^{At}y_0$. Assume that the Jordan canonical form of A has only Jordan blocks corresponding to polynomials of degree 1. Use this Jordan canonical form to compute e^{At}.