Rings and modules - Problem set 5

To be solved on Friday 10.11

Problem 1. (After Chapter 11.3.) Let R be a PID. Let $r, s \in R$ and let $d = \gcd(r, s)$. Show that $(r) + (s) = (d)$.

Problem 2. (After Chapter 20.3) (Exercise 20.3.1 in the book.) Obtain the Smith normal form and rank for the following matrices over a PID R:

(a) $\begin{pmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{pmatrix}$, where $R = \mathbb{Z}$.

(b) $\begin{pmatrix} -X-3 & 2 & 0 \\ 1 & -X & 1 \\ 1 & -3 & -X-2 \end{pmatrix}$, where $R = \mathbb{Q}[X]$.

Problem 3. (After Chapter 20.3) (Exercise 20.3.3 in the book.) Find the rank of the subgroup of $\mathbb{Z}^4$ generated by each of the following lists of elements.

(a) $(3, 6, 9, 0), (-4, -8, -12, 0)$.

(b) $(2, 3, 1, 4), (1, 2, 3, 0), (1, 1, 1, 4)$.

(c) $(-1, 2, 0, 0), (2, -3, 1, 0), (1, 1, 1, 1)$.

Problem 4. (After Chapter 20.3) (Exercise 20.3.2 in the book.) Find the invariant factors of the following matrix over $\mathbb{Q}[X]$: $\begin{pmatrix} 5-X & 1 & -2 & 4 \\ 0 & 5-X & 2 & 2 \\ 0 & 0 & 5-X & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Problem 5. (After Chapter 14.3.) (Exam November 2005, Problem 1.) Let q be a fixed non-zero element in $\mathbb{C}$, the set of complex numbers. Define the subset R_q of the ring of 4×4-matrices over $\mathbb{C}$ by

$$R_q = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & -qb & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

(a) Show that R_q is a unital ring.

(b) For which q in $\mathbb{C}$ is R_q a commutative ring?

(c) For a given element α in $\mathbb{C}$ define the subset

$$I_\alpha = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ \alpha x & 0 & 0 & 0 \\ y & \alpha x & -qx & 0 \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$$

of R_q. Show that I_α is a left ideal in R_q for all $\alpha \in \mathbb{C}$.

(d) Show that each of the left ideals I_α is generated by one element as a left ideal. Show that $I_\alpha \cong R_q / I_{\alpha q}$ as left R_q-modules.

Problem 6. (After Chapter 10.2.)

(a) Let R be a unital ring. An idempotent e of R is called *central* if $e \in Z(R)$. We say that R is *connected* if there exist no nonzero unital rings T_1, T_2 such that $R \cong T_1 \times T_2$. Show that the following are equivalent

  (i) R is connected.

  (ii) the only central idempotents of R are 0 and 1.

  (Hint: to show that (i) implies (ii), assume instead that there exists a central idempotent $e \in R$ with $e \notin \{0, 1\}$ and consider the ring $eR \times (1-e)R$.)

(b) Let $R_1, \ldots, R_p$ be unital rings. Show that the following are equivalent.

  (i) The ring $R_1 \times \cdots \times R_p$ has exactly 2^p central idempotents.

  (ii) $R_1, \ldots, R_p$ are all connected rings.

  (Hint: show first that $R_1 \times \cdots \times R_p$ has at least 2^p idempotents irrespectively of $R_1, \ldots, R_p$ being connected.)

(c) Let $R_1, \ldots, R_p$ and $S_1, \ldots, S_q$ be connected unital rings. Show that if there is a ring isomorphism

$$R_1 \times \cdots \times R_p \cong S_1 \times \cdots \times S_q,$$

then $p = q$.

(d) Let D be a division ring and $n > 0$ a positive integer. Show that the ring $M_n(D)$ is connected. *(Hint: use Problem 8 from Problem Set 1 to describe the center of $M_n(R)$, using the obvious generalization from the case 2 to the case n.)*

(e) Let $D_1, \ldots, D_p$ and $D'_1, \ldots, D'_q$ be division rings and let $n_1, \ldots, n_p$ and $k_1, \ldots, k_q$ be positive integers. Show that if there is a ring isomorphism

$$M_{n_1}(D_1) \times \cdots \times M_{n_p}(D_p) \cong M_{k_1}(D'_1) \times \cdots \times M_{k_q}(D'_q),$$

then $p = q$.

Problem 7. (After Chapter 19.2.) (Exam November 2005, Problem 4.) Let R be a unital ring and let M be a noetherian left R-module. Show that any surjective homomorphism of R-modules $f : M \to M$ is an isomorphism. (Hint: Consider the chain $\ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \cdots$ of submodules of M).

Problem 8. (After Chapter 19.3.) (Exam November 2005, Problem 3.) Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[X]$ the polynomial ring over $\mathbb{C}$ in one variable X. Let $\alpha \in \mathbb{C}$ be a complex number.

(a) Show that the map $\phi_\alpha : \mathbb{C}[X] \to \mathbb{C}$ defined by $\phi_\alpha(f(X)) = f(\alpha)$ is a surjective ring homomorphism, and use this to show that the ideal generated by $X - \alpha$ is a maximal ideal in $\mathbb{C}[X]$.

(b) For which $n \geq 1$ is the ring

$$\begin{pmatrix} \frac{\mathbb{C}[X]}{((X-\alpha)^n)} & \frac{\mathbb{C}[X]}{((X-\alpha)^n)} \\ \frac{\mathbb{C}[X]}{((X-\alpha)^n)} & \frac{\mathbb{C}[X]}{((X-\alpha)^n)} \end{pmatrix}$$

semisimple?

Problem 9. (After Chapter 19.3.) (Exam December 2015, Problem 2.) Let $\Lambda = \left\{ \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_6 \right\} \subseteq M_3(\mathbb{Z}_6)$ be the ring of 3×3 matrices over $\mathbb{Z}_6$.

(a) Prove that Λ is a commutative subring of $M_3(\mathbb{Z}_6)$, the ring of 3×3-matrices over $\mathbb{Z}_6$.

(b) Define $\Psi : \Lambda \to \mathbb{Z}_6$ by $\Psi\left(\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\right) = a + b + c$. Prove that Ψ is a surjective ring homomorphism and find a set of generators for the kernel of Ψ.

(c) How many maximal ideals in Λ contain the kernel of Ψ? You have to give an argument for your answer.

(d) Is Λ a semisimple ring? You have to give an argument for your answer. *(Hint: find how many idempotents Λ has.)*

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 10. (After Chapter 19.3.) Let K be a field. A K-algebra Λ is called a *division algebra* if Λ is a division ring as a ring, that is $\Lambda \neq 0$ is a unital ring and for every nonzero $r \in \Lambda$ there exists a multiplicative inverse $r^{-1} \in \Lambda$.

(a) Show that if Λ is a semisimple finite-dimensional unital algebra over K, then there exist finite-dimensional division algebras $D_1, \ldots, D_k$ over K and positive integers $n_1, \ldots, n_k$ such that

$$\Lambda \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

(b) Show that if Λ is a semisimple finite-dimensional algebra over $\mathbb{C}$, then there exist positive integers $n_1, \ldots, n_k$ such that

$$\Lambda \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C}).$$

(Hint: use the fundamental theorem of algebra.)

Problem 11. (After Chapter 19.3.) The aim of this problem is to prove Maschke's theorem. Let F be a field and G be a finite group such that the characteristic of F does not divide the order of G. Recall that

$$F[G] = \{f : G \to F \mid f(g) = 0 \text{ for all but finitely many } g \in G\}$$

with addition given by

$$(f + h)(g) = f(g) + h(g)$$

and multiplication given by

$$(fh)(g) = \sum_{g=g_1 g_2} f(g_1)h(g_2),$$

where $f, h \in F[G]$ and $g \in G$. Let M be a finitely generated left $F[G]$-module where $F[G]$ is the group algebra.

(a) Show that $F[G]$ is isomorphic to the ring

$$FG = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \lambda g \in F \right\},$$

with addition given by

$$\left(\sum_{g \in G} \lambda_g g\right) + \left(\sum_{g \in G} \mu_g g\right) = \sum_{g \in G} (\lambda_g + \mu_g) g$$

with $0_{FG} = \sum_{g \in G} 0g$ and multiplication given by

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{g \in G} \mu_g g\right) = \sum_{g \in G}\left(\sum_{g_1 g_2 = g \in G} \lambda_{g_1}\mu_{g_2}\right) g,$$

with $1_{FG} = e = e_g$. Conclude that $F[G]$ is a finite-dimensional vector space over F. Use this description of $F[G]$ for the rest of the problem.

(b) Show that M is a finite-dimensional vector space over F.

(c) Show that M is a left artinian $F[G]$-module. Conclude that either $M = 0$ or M has a simple submodule.

(d) Let $f \in \mathrm{End}_F(M)$. Define $\tilde{f} : M \to M$ via

$$\tilde{f}(m) = \frac{1}{|G|} \sum_{g \in G} g f(g^{-1} m).$$

Show that $\tilde{f} \in \mathrm{End}_{F[G]}(M)$.

(e) Assume that there exists a simple submodule $S \subseteq M$. In particular, S is a subspace of M and so there exists a subspace $N \subseteq M$ such that $M = S \oplus N$ (as vector spaces). Let $\pi : M \to S$ and $\iota : S \to M$ be the canonical projection and inclusion maps. Show that if $f = \iota \circ \pi$, then $\operatorname{Im} \tilde{f} = S$ and $\tilde{f}^2 = \tilde{f}$.

(f) Show that there if there exists a simple submodule $S \subseteq M$, then there is an isomorphism of left $F[G]$-modules

$$M \cong \operatorname{Im} \tilde{f} \oplus \operatorname{Im}(1_M - \tilde{f}).$$

Conclude that every finitely generated $F[G]$-module is semisimple.