Rings and modules - Problem set 4 solutions

Solved on Tuesday 31.10

Problem 1. (After Chapter 14.5.) Show that $\{2, 3\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module, but no subset of $\{2, 3\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}$.

Solution. Let $n \in \mathbb{Z}$ be an integer. If n is even, then $n = 2m$ for some $m \in \mathbb{Z}$. If n is odd, then $n = 2k + 1 = 2(k-1) + 3$ for some $k \in \mathbb{Z}$. Hence $n \in (2, 3)$ and so $\{2, 3\}$ generates $\mathbb{Z}$.

The subsets of $\{2, 3\}$ are $\varnothing$, $\{2\}$, $\{3\}$, $\{2, 3\}$. Clearly $(\varnothing) = 0 \neq \mathbb{Z}$, $(2) = \{2m \mid m \in \mathbb{Z}\} \neq \mathbb{Z}$, $(3) = \{3m \mid m \in \mathbb{Z}\} \neq \mathbb{Z}$ and so none of the sets $\varnothing$, $\{2\}$, $\{3\}$ generate $\mathbb{Z}$. Hence none of the sets $\varnothing$, $\{2\}$ and $\{3\}$ is a basis. On the other hand, the set $\{2, 3\}$ is not linearly independent since

$$3 \cdot 2 + (-2) \cdot 3 = 0$$

and $(3, -2) \neq (0, 0)$. Hence $\{2, 3\}$ is not a basis either and so no subset of $\{2, 3\}$ is a basis.

Problem 2. (After Chapter 14.4.) Let D be a division ring and $n > 0$ a positive integer. Let $R = M_n(D)$ and let $e_{kk} \in R$ be the matrix with 1 in position (k, k) and 0 everywhere else.

(a) Show that D^{op} is a division ring.

(b) Show that ${}_RR = Re_{11} \oplus \cdots \oplus Re_{kk}$ and so R is a left semisimple ring.

Solution.

(a) Let us write $\cdot_{D^{\text{op}}}$ for the multiplication in D^{op}, and let us write $\cdot_D$ for the multiplication in D. Since D is a division ring, we have that $1 \in D = D^{\text{op}}$. Then for any $a \in D$ we have

$$a \cdot_{D^{\text{op}}} 1 = 1 \cdot_D a = a = a \cdot_D 1 = 1 \cdot_{D^{\text{op}}} a,$$

which shows that 1 acts as the multiplicative identity in D^{op} as well. Hence D^{op} is unital. Then let $a \in D^{\text{op}} \setminus \{0\}$. We have that $a^{-1} \in D$ and

$$a \cdot_{D^{\text{op}}} a^{-1} = a^{-1} \cdot_D a = 1 = a \cdot_D a^{-1} = a^{-1} \cdot_{D^{\text{op}}} a,$$

which shows that a^{-1} is the multiplicative inverse of a in D^{op} as well. This shows that D^{op} is a division ring.

(b) We have that

$$Re_{kk} = \left\{ \left. \begin{pmatrix} 0 & \cdots & 0 & r_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & r_{nk} & 0 & \cdots & 0 \end{pmatrix} \right| r_{1k}, \ldots, r_{nk} \in R \right\}.$$

Then if $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in M_n(R)$, we have that

$$A = \underbrace{\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{pmatrix}}_{\in Re_{11}} + \cdots + \underbrace{\begin{pmatrix} 0 & \cdots & 0 & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & 0 & \cdots & 0 \end{pmatrix}}_{\in Re_{kk}} + \cdots + \underbrace{\begin{pmatrix} 0 & \cdots & 0 & a_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}}_{\in Re_{nn}} \in Re_{11} + \cdots + Re_{nn},$$

which shows that ${}_RR = Re_{11} + \cdots + Re_{nn}$. To show that the sum is direct, let $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in Re_{11} + \cdots + Re_{nn}$. Then

$$0 = A_1 + \cdots A_n$$

for some $A_k = \begin{pmatrix} 0 & \cdots & 0 & r_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & r_{nk} & 0 & \cdots & 0 \end{pmatrix} \in Re_{kk}$. Hence

$$\begin{aligned} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} &= \begin{pmatrix} r_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ r_{n1} & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & 0 & r_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & r_{nk} & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & 0 & r_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{pmatrix} \\ &= \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}, \end{aligned}$$

which gives that $r_{ij} = 0$ for all $1 \leq i, j \leq n$. Thus $A_k = 0$ for all $k = 1, \ldots, n$ which shows that the sum is direct by Theorem 9.12(2).

Problem 3. (After Chapter 19.2.) (Exercise 19.2.3 in the book.) Show that every principal left ideal unital ring is left noetherian.

Solution. Let R be a principal left ideal unital ring. Hence every left ideal of R is generated by a single element. In particular, every left ideal of R is finitely generated. By Corollary 13.6 we obtain that R is a left noetherian ring.

Problem 4. (After Chapter 14.3.) Let R be a ring that is not unital. Let $\tilde{R} = R \times \mathbb{Z}$. For $(r, n), (s, k) \in \tilde{R}$ define

$$(r, n) + (s, k) := (r + s, n + k), \quad (r, n)(s, k) := (rs + kr + ns, nk).$$

(a) Show that $\tilde{R}$ is a unital ring with $1_{\tilde{R}} = (0, 1)$.

(b) Let M be a left R-module. For $m \in M$ and for $(r, n) \in \tilde{R}$ define

$$(r, n)m := rm + nm.$$

Show that this makes M a left $\tilde{R}$-module. Also show that if $f : {}_RM \to {}_RN$ is an homomorphism of left R-modules, then $f : {}_{\tilde{R}}M \to {}_{\tilde{R}}N$ is also a homomorphism of left $\tilde{R}$-modules.

(c) Let $\tilde{M}$ be a left $\tilde{R}$-module. For $\tilde{m} \in \tilde{M}$ and for $r \in R$ define

$$r\tilde{m} := (r, 0)\tilde{m}.$$

Show that this makes $\tilde{M}$ a left R-module. Also show that if $\tilde{f} : {}_{\tilde{R}}\tilde{M} \to {}_{\tilde{R}}\tilde{N}$ is a homomorphism of left $\tilde{R}$-modules, then $\tilde{f} : {}_RM \to {}_RN$ is a homomorphism of left R-modules.

(d) Show that the two constructions in (b) and (c) are inverse to each other.

Thus this problem shows that studying left R-modules and homomorphisms of left R-modules is the same as studying $\tilde{R}$-modules and homomorphisms of left $\tilde{R}$-modules, motivating the fact that we focus on unital rings.

Solution.

(a) Let $(r, n), (s, k), (t, l) \in \tilde{R}$. We have that

$$\begin{aligned} ((r, n) + (s, k)) + (t, l) &= (r + s, k + n) + (t, l) = ((r + s) + t, (k + n) + l) = (r + (s + t), k + (n + l)) \\ &= (r, k) + (s + t, n + l) = (r, k) + ((s, n) + (t, l)), \\ (r, n) + (s, k) &= (r + s, n + k) = (s + r, k + n) = (s, k) + (r, n), \\ (0, 0) + (r, n) &= (0 + r, 0 + n) = (r, n), \\ (r, n) + (-r, -n) &= (r + (-r), n + (-n)) = (0, 0), \end{aligned}$$

which shows that $(\tilde{R}, +)$ is an abelian group. Next, we have

$$\begin{aligned}((r,n)(s,k))(t,l) &= (rs + kr + ns, nk)(t,l) \\ &= ((rs + kr + ns)t + l(rs + kr + ns) + (nk)t, (nk)l) \\ &= (rst + krt + nst + lrs + lkr + lns + nkt, nkl) \\ &= ((rst + lrs + krt) + lkr + (nst + lns + nkt), nkl) \\ &= (r(st + ls + kt) + (kl)r + n(st + ls + kt), n(kl)) \\ &= (r,n)(st + ls + kt, kl) \\ &= (r,n)((s,k)(t,l)),\end{aligned}$$

which shows that $(\tilde{R}, \cdot)$ is a semigroup. Next, we have

$$\begin{aligned}(r,n)((s,k) + (t,l)) &= (r,n)(s + t, k + l) \\ &= (r(s+t) + (k+l)r + n(s+t), n(k+l)) \\ &= (rs + rt + kr + lr + ns + nt, nk + nl) \\ &= ((rs + kr + ns) + (rt + lr + nt), nk + nl) \\ &= (rs + kr + ns, nk) + (rt + lr + nt, nl) \\ &= (r,n)(s,k) + (r,n)(t,l),\end{aligned}$$

which shows that $(r,n)((s,k) + (t,l)) = (r,n)(s,k) + (r,n)(t,l)$. Similarly we can show that $((r,n) + (s,k))(t,l) = (r,n)(t,l) + (s,k)(t,l)$. This shows that $\tilde{R}$ is a ring. To see that it is unital, we have

$$\begin{aligned}(r,n)(0,1) &= (r0 + 1r + 0n, n1) = (0 + r + 0, n) = (r,n) \\ (0,1)(r,n) &= (0r + n0 + 1r, 1n) = (0 + 0 + r, n) = (r,n)\end{aligned}$$

and so $1_{\tilde{R}} = (0,1)$.

(b) M is an additive group by definition. Let $(r_1,n_1),(r_2,n_2),(r,n) \in \tilde{R}$ and $m_1,m_2,m \in M$. Then

$$\begin{aligned}(r,n)(m_1 + m_2) &= r(m_1 + m_2) + n(m_1 + m_2) = rm_1 + rm_2 + nm_1 + nm_2 \\ &= rm_1 + nm_1 + rm_2 + nm_2 = (r,n)m_1 + (r,n)m_2 \\ ((r_1,n_1) + (r_2,n_2))m &= (r_1 + r_2, n_1 + n_2)m = (r_1 + r_2)m + (n_1 + n_2)m = r_1m + r_2m + n_1m + n_2m \\ &= r_1m + n_1m + r_2m + n_2m = (r_1,n_1)m + (r_2,n_2)m \\ ((r_1,n_1)(r_2,n_2))m &= (r_1r_2 + n_2r_1 + n_1r_2, n_1n_2)m = (r_1r_2 + n_2r_1 + n_1r_2)m + (n_1n_2)m \\ &= (r_1r_2m + r_1n_2m) + (n_1r_2m + n_1n_2m) = r_1(r_2m + n_2m) + n_1(r_2m + n_2m) \\ &= (r_1,n_1)(r_2m + n_2m) = (r_1,n_1)((r_2,n_2)m) \\ (0,1)m &= 0m + 1m = 0 + m = m,\end{aligned}$$

and so M is a left $\tilde{R}$-module. Now let $f : {}_RM \to {}_RN$ be a homomorphism of left R-modules. Let $m_1, m_2, m \in M$ and $(r,n) \in \tilde{R}$. Then clearly

$$f(m_1 + m_2) = f(m_1) + f(m_2),$$

since f is a homomorphism of left R-modules. Moreover, using the fact that f is a homomorphism of left R-modules and using Proposition 10.2 we obtain

$$f((r,n)m) = f(rm + nm) = f(rm) + f(nm) = rf(m) + nf(m) = (r,n)f(m),$$

showing that f is a homomorphism of left $\tilde{R}$-modules.

(c) $\tilde{M}$ is an additive group by definition. Let $r_1, r_2, r \in R$ and $\tilde{m}_1, \tilde{m}_2, \tilde{m} \in M$. Then

$$\begin{aligned}r(\tilde{m}_1 + \tilde{m}_2) &= (r,0)(\tilde{m}_1 + \tilde{m}_2) = (r,0)\tilde{m}_1 + (r,0)\tilde{m}_2 = r\tilde{m}_1 + r\tilde{m}_2 \\ (r_1 + r_2)\tilde{m} &= (r_1 + r_2, 0)\tilde{m} = ((r_1,0) + (r_2,0))\tilde{m} = (r_1,0)\tilde{m} + (r_2,0)\tilde{m} = r_1\tilde{m} + r_2\tilde{m} \\ (r_1r_2)\tilde{m} &= (r_1r_2, 0)\tilde{m} = ((r_1,0)(r_2,0))\tilde{m} = (r_1,0)((r_2,0)\tilde{m}) = (r_1,0)(r_2\tilde{m}) = r_1(r_2\tilde{m}),\end{aligned}$$

and so $\tilde{M}$ is a left R-module. Now let $f : {}_{\tilde{R}}M \to {}_{\tilde{R}}N$ be a homomorphism of left $\tilde{R}$-modules. Let $\tilde{m}_1, \tilde{m}_2, \tilde{m} \in M$ and $r \in R$. Then clearly

$$f(\tilde{m}_1 + \tilde{m}_2) = f(\tilde{m}_1) + f(\tilde{m}_2),$$

since f is a homomorphism of left $\tilde{R}$-modules. Moreover, using the fact that f is a homomorphism of left $\tilde{R}$-modules we obtain

$$f(r\tilde{m}) = f((r,0)\tilde{m}) = (r,0)f(\tilde{m}) = rf(\tilde{m}),$$

showing that f is a homomorphism of left R-modules.

(d) Let $r \in R$ and $(r,n) \in \tilde{R}$. First let M be a left R-module and let $m \in M$. Write $r \cdot_R m$ for the left R-scalar multiplication of M and write $(r,n) \cdot_{\tilde{R}} m$ for the left $\tilde{R}$-scalar multiplication of M. We have

$$(r,0) \cdot_{\tilde{R}} m = r \cdot_R m + 0 \cdot_R m = r \cdot_R m,$$

and so $(r,0) \cdot_{\tilde{R}} m = r \cdot_R m$.

Now let $\tilde{M}$ be a left $\tilde{R}$-module and let $\tilde{m} \in \tilde{M}$. Write $(r,n) \cdot_{\tilde{R}} \tilde{m}$ for the left $\tilde{R}$-scalar multiplication of $\tilde{M}$ and write $r \cdot_R \tilde{m}$ for the left R-scalar multiplication of $\tilde{M}$. Since $1_{\tilde{R}} = (0,1)$, we have that $n\tilde{m} = (0,n) \cdot_{\tilde{R}} \tilde{m}$. Then we have

$$r \cdot_R \tilde{m} + n\tilde{m} = (r,0) \cdot_{\tilde{R}} \tilde{m} + (0,n) \cdot_{\tilde{R}} \tilde{m} = ((r,0) + (0,n)) \cdot_{\tilde{R}} \tilde{m} = (r,n) \cdot_{\tilde{R}} \tilde{m},$$

and so $r \cdot_R \tilde{m} + n\tilde{m} = (r,n) \cdot_{\tilde{R}} \tilde{m}$.

Problem 5. (After Chapter 10.2.) Let R and S be rings and $n \geq 1$ a positive integer.

(a) Show that the rings $(R \times S)^{\mathrm{op}}$ and $R^{\mathrm{op}} \times S^{\mathrm{op}}$ are isomorphicm.

(b) Show that the rings $(M_n(R))^{\mathrm{op}}$ and $M_n(R^{\mathrm{op}})$ are isomorphic.

Solution.

(a) Since R and R^{op} are the same as sets and S and S^{op} are also the same as sets, we obtain that $(R \times S)^{\mathrm{op}}$ and $R^{\mathrm{op}} \times S^{\mathrm{op}}$ are the same as sets (and equal to $R \times S$). We define a map $g : (R \times S)^{\mathrm{op}} \to R^{\mathrm{op}} \times S^{\mathrm{op}}$ by setting g to be the identity. Then g is clearly bijective as the identity map is bijective. It is enough to show that g is a ring homomorphism. Let $(r_1, s_1), (r_2, s_2) \in (R \times S)^{\mathrm{op}}$. Then

$$g((r_1,s_1)+(r_2,s_2)) = g((r_1+r_2, s_1+s_2)) = (r_1+r_2, s_1+s_2) = (r_1,s_1)+(r_2,s_2) = g((r_1,s_1))+g((r_2,s_2)),$$

and

$$\begin{aligned} g((r_1,s_1) \cdot_{(R\times S)^{\mathrm{op}}} (r_2,s_2)) &= g((r_2,s_2) \cdot_{R\times S} (r_1,s_1)) = g((r_2 \cdot_R r_1, s_2 \cdot_S s_1)) \\ &= (r_2 \cdot_R r_1, s_2 \cdot_S s_1) = (r_1 \cdot_{R^{\mathrm{op}}} r_2, s_1 \cdot_{S^{\mathrm{op}}} s_2) = (r_1,s_1) \cdot_{R^{\mathrm{op}}\times S^{\mathrm{op}}} (r_2,s_2) \\ &= g((r_1,s_2)) \cdot_{R^{\mathrm{op}}\times S^{\mathrm{op}}} g((r_2,s_2)), \end{aligned}$$

which shows that g is a ring homomorphism, as required.

(b) By definition, the elements of R and R^{op} are the same and the elements of $M_n(R)$ and $(M_n(R))^{\mathrm{op}}$ are the same. It follows that the elements of $(M_n(R))^{\mathrm{op}}$ and $M_n(R^{\mathrm{op}})$ are the same. Set

$$A = M_n(R), B = M_n(R^{\mathrm{op}}), C = (M_n(R))^{\mathrm{op}}.$$

Our aim is to show that the rings B and C are isomorphic. We define $f : C \to B$ to be the transpose map, that is if $X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in C$, then

$$f(X) = X^T = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}.$$

We claim that f is a ring isomorphism. Clearly the identity map is bijective with inverse $g : B \to C$ the map $g(Y) = Y^T$, and so it remains to show that it is a ring homomorphism. Let $X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$ and $Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}$ be two elements of $C = (M_n(R))^{\text{op}}$. Then

$$\begin{aligned} f(X+Y) &= f\left(\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} + \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}\right) = f\left(\begin{pmatrix} x_{11}+y_{11} & \cdots & x_{1n}+y_{1n} \\ \vdots & & \vdots \\ x_{n1}+y_{n1} & \cdots & x_{nn}+y_{nn} \end{pmatrix}\right) \\ &= \begin{pmatrix} x_{11}+y_{11} & \cdots & x_{n1}+y_{n1} \\ \vdots & & \vdots \\ x_{1n}+y_{1n} & \cdots & x_{nn}+y_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} + \begin{pmatrix} y_{11} & \cdots & y_{n1} \\ \vdots & & \vdots \\ y_{1n} & \cdots & y_{nn} \end{pmatrix} = f(X) + f(Y), \end{aligned}$$

and

$$\begin{aligned} f(X \cdot_C Y) &= f\left(\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \cdot_C \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}\right) = f\left(\begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix} \cdot_A \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} y_{11} \cdot_R x_{11} + \cdots + y_{1n} \cdot_R x_{n1} & \cdots & y_{11} \cdot_R x_{1n} + \cdots + y_{1n} \cdot_R x_{nn} \\ \vdots & & \vdots \\ y_{n1} \cdot_R x_{11} + \cdots + y_{nn} \cdot_R x_{n1} & \cdots & y_{n1} \cdot_R x_{1n} + \cdots y_{nn} \cdot_R x_{nn} \end{pmatrix}\right) \\ &= \begin{pmatrix} y_{11} \cdot_R x_{11} + \cdots + y_{1n} \cdot_R x_{n1} & \cdots & y_{n1} \cdot_R x_{11} + \cdots + y_{nn} \cdot_R x_{n1} \\ \vdots & & \vdots \\ y_{11} \cdot_R x_{1n} + \cdots + y_{1n} \cdot_R x_{nn} & \cdots & y_{n1} \cdot_R x_{1n} + \cdots y_{nn} \cdot_R x_{nn} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} \cdot_{R^{\text{op}}} y_{11} + \cdots + x_{n1} \cdot_{R^{\text{op}}} y_{1n} & \cdots & x_{11} \cdot_{R^{\text{op}}} y_{n1} + \cdots + x_{n1} \cdot_{R^{\text{op}}} y_{nn} \\ \vdots & & \vdots \\ x_{1n} \cdot_{R^{\text{op}}} y_{11} + \cdots + x_{nn} \cdot_{R^{\text{op}}} y_{1n} & \cdots & x_{1n} \cdot_{R^{\text{op}}} y_{n1} + \cdots x_{nn} \cdot_{R^{\text{op}}} y_{nn} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \cdot_B \begin{pmatrix} y_{11} & \cdots & y_{n1} \\ \vdots & & \vdots \\ y_{1n} & \cdots & y_{nn} \end{pmatrix} = f(X) \cdot_B f(Y), \end{aligned}$$

which shows that f is indeed a ring homomorphism, as required.

Problem 6. (After Chapter 19.3.) (Exercise 19.3.1 in the book.) Let R be a left artinian unital ring with no nonzero nilpotent ideals. Show that for each two-sided ideal I of R, R/I is also left artinian with no nonzero nilpotent ideals. *(Hint: use Proposition 14.5 and Wedderburn–Artin theorem.)*

Solution. Let R be a left artinian ring with no nonzero nilpotent ideals. By Proposition 14.5 we have that this is equivalent to R being a left semisimple ring. Hence by the Wedderburn–Artin theorem we have that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k),$$

where $n_1, \ldots, n_k$ are positive integers and $D_1, \ldots, D_k$ are division rings. By Theorem 3.4(3) in the notes, the only two-sided ideals of $M_{n_i}(D_i)$ are the trivial ideals. Hence an ideal I of R is isomorphic to an ideal of $M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ of the form $I_1 \times \cdots \times I_k$, where I_i is either 0 or $M_{n_i}(D_i)$. Then

$$R/I \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)/(I_1 \times \cdots \times I_k) \cong (M_{n_1}(D_1)/I_1) \times \cdots \times (M_{n_k}(D_k)/I_k)$$

and $M_{n_i}(D_i)/I_i$ is isomorphic to either 0 or $M_{n_i}(D_i)$. Hence R/I is isomorphic to a product of matrix rings over division rings. By the Wedderburn-Artin theorem, R/I is a left semisimple ring. By Proposition 14.5, we conclude that R/I is a artinian with no nonzero nilpotent ideals.

Problem 7. (After Chapter 10.3.) (Exam December 2009, Problem 4.) Let A be a left ideal in a unital ring R and assume that $A = Aa$ for some $a \neq 0$ in A.

(a) Show that there is some $e \in A$ where $ea \neq 0$ and $(e^2 - e)a = 0$.

(b) Let $B = \{x \in A \mid xa = 0\}$. Show that B is a left ideal in R.

(c) Assume further that the left ideal A is a minimal left ideal. Show that the element e from (a) is then an idempotent element.

(You can use (a) to show (b) even if you do not show (a), and you can use (a) and (b) to show (c).)

Solution.

(a) Since $a \in A = Aa$, there exists some $e \in A$ such that $a = ea$. Since $a \neq 0$, we have $ea \neq 0$. Moreover, $(e^2 - e)a = e^2a - ea = e(ea) - ea = ea - ea = 0$, as required.

(b) By part (a) we have that $e^2 - e \in B$ and so $B \neq \varnothing$. Let $x, y \in B$ and $r \in R$. Since $x, y \in B$, we have that $xa = 0$ and $ya = 0$. Then

$$(x - y)a = xa - ya = 0 + 0 = 0,$$

and so $x - y \in B$. Moreover, we have

$$(rx)a = r(xa) = r0 = 0,$$

and so $rx \in B$. This shows that B is a left ideal.

(c) By definition, we have that $B \subseteq A$. Since A is a minimal left ideal and B is a left ideal contained in A, we have that $B = 0$ or $B = A$. By part (a) we have that $ea \neq 0$ and so $e \notin B$ while $e \in A$. Hence $B \neq A$ and so $B = 0$. Then $e^2 - e \in B = 0$ gives $e^2 - e = 0$ or $e^2 = e$. Thus e is an idempotent element.

Problem 8. (After Chapter 14.5.) Let R be a unital ring. The aim of this problem is to show that every left R-module is the quotient of a free R-module.

(a) Let I be a set and let

$$R^I = \bigoplus_{i \in I} R = \{(r_i)_{i \in I} \mid r_i \in R \text{ and finitely many } r_i \neq 0\}.$$

Show that R^I is a free left R-module with a basis indexed by I.

(b) Let M be a left R-module. Show that there is a free R-module F and a submodule $K \subseteq F$ such that $M \cong F/K$. Show that if M is finitely generated, then F can also be taken to be finitely generated. *(Hint: use part (a) with $I = M$.)*

Solution.

(a) For $i \in I$ let $e_i = (r_j)_{j \in I}$ where

$$r_j = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

We claim that $\{e_i\}_{i \in I}$ is a basis of R^I. Let $r = (r_i)_{i \in I} \in M$. Then only finitely many of the r_i are nonzero and so

$$r = (r_i) = \sum_{\substack{i \in I \\ r_i \neq 0}} r_i e_i$$

and so $\{e_i\}_{i \in I}$ generates R^I. Now let $I' \subseteq I$ be a finite subset of I and assume that

$$\sum_{i \in I'} r_i e_i = 0 = (0)_{i \in I}.$$

Then $\sum_{i\in I'} r_i e_i = (u_i) \in R^I$ where

$$u_i = \begin{cases} r_i, & \text{if } i \in I', \\ 0, & \text{if } i \notin I'. \end{cases}$$

Then $(u_i) = 0$ implies that $u_i = 0$ for all $i \in I$ and so $r_i = 0$ for all $i \in I'$. We conclude that $\{e_i\}_{i\in I'}$ is an R-linearly independent set and so they form an R-basis of R^I.

(b) Let F be the free module R^M as in part (a), that is

$$R^M = \bigoplus_{m\in M} R = \{(r_m)_{m\in M} \mid r_m \in R \text{ and finitely many } r, \neq 0\}.$$

Define a map $f : F \to M$ by $f((r_m)_{m\in M}) = \sum_{m\in M} r_m m$. Notice that since there are only finitely many non-zero terms in $(r_m)_{m\in M}$ this is a well defined map, since the sum $\sum_{m\in M} r_m m$ is a finite sum. It is straightforward to check that f is a homomorphism of left R-modules. Moreover, for any $m \in M$ we have that $f(e_m) = 1m = m$ and so f is surjective. By the first isomorphism theorem for modules we obtain that $M \cong F/\operatorname{Ker} f$. Since $K = \operatorname{Ker} f$ is a submodule of F, the claim follows.

Now assume that M is finitely generated and let $\{m_1, \dots, m_k\}$ be a generating set of M. Then we set

$$F = R^{\{1,\dots,k\}};$$

Then F is finitely generated, since $\{e_1, \dots, e_k\}$ is a basis of F by part (a). Now define $g : F \to M$ via

$$g\left(\sum_{i=1}^{k} r_i e_i\right) = \sum_{i=1}^{k} m_i.$$

Again, it is easy to see that g is a homomorphism of left R-modules. Then $\operatorname{Im} g$ is a submodule of M containing $\{m_1, \dots, m_k\}$. Since

$$M = (m_1, \dots, m_k) = \operatorname{Im} g \subseteq M,$$

we conclude that $\operatorname{Im} g = M$ and so g is surjective. Again by the first isomorphism theorem for modules we obtain that $M \cong F/\operatorname{Ker} g$, with F finitely generated.

Problem 9. (After Chapter 19.3.) Let $R_1, \dots, R_m$ be unital left semisimple rings. Show that $R_1 \times \cdots \times R_m$ is a left semisimple ring, without using the Wedderburn–Artin theorem.

Solution. It is enough to show the result for $m = 2$ since the claim for $m > 2$ follows by induction. Since R_1 is left semisimple, we have that

$${}_{R_1}R_1 = S_1 + \cdots + S_n$$

for some simple left R_1-modules $S_1, \dots, S_n$. Similarly we have that

$${}_{R_2}R_2 = T_1 + \cdots + T_k$$

for some simple left R_2-modules $T_1, \dots, T_k$. We set

$$S_1' = \{(s, 0) \mid s \in S_1\} = (S_1, 0).$$

We claim that S_1' is a simple left $R_1 \times R_2$ module via defining

$$(r_1, r_2)(s, 0) = (r_1 s, 0)$$

for any $(r_1, r_2) \in R_1 \times R_2$ and $(s, 0) \in S_1'$. It is a direct verification that this defines an $R_1 \times R_2$-module structure on S_1'. To see that S_1' is simple, first note that $S_1' \neq 0$ since S_1 is a simple R_1-module. Now let $U \subseteq S_1'$ be a submodule and assume that $U \neq 0$. Let $u \in U \setminus \{0\}$. Then

$$(R_1 \times R_2)u = \{(r_1, r_2)(u, 0) \mid (r_1, r_2) \in R_1 \times R_2\} = \{(r_1 u, 0) \mid (r_1, r_2) \in R_1 \times R_2\} = (R_1 u, 0).$$

But R_1u is a R_1-submodule of S_1 with $R_1u \neq 0$ (since $u \neq 0$). Since S_1 is simple, we conclude that $R_1u = S_1$. Then we have

$$S_1' = (S_1, 0) = (R_1u, 0) = (R_1 \times R_2)u \subseteq U \subseteq S_1',$$

which shows that $S_1' = U$. Hence S_1 is a simple $R_1 \times R_2$-module.

Now for $1 \leq i \leq n$ and for $1 \leq j \leq k$ we set $S_i' = (S_1, 0)$ and $T_j' = (0, T_j)$. Similarly we obtain that S_i' and T_j' are simple $R_1 \times R_2$-modules. We claim that, as a left $R_1 \times R_2$-module, we have

$${}_{R_1 \times R_2}(R_1 \times R_2) = S_1' + \cdots S_n' + T_1' + \cdots + T_k'.$$

Indeed, let $(r_1, r_2) \in R_1 \times R_2$. Then $r_1 = s_1 + \cdots + s_n$ for some $s_i \in S_i$ and $r_2 = t_1 + \cdots + t_k$ for some $t_j \in T_j$. Hence

$$(r_1, r_2) = (s_1 + \cdots + s_n, t_1 + \cdots + t_k) = \underbrace{(s_1, 0)}_{\in S_1'} + \cdots + \underbrace{(s_n, 0)}_{\in S_n'} + \underbrace{(0, t_1)}_{\in T_1'} + \cdots + \underbrace{(0, t_k)}_{\in T_k'} \in S_1' + \cdots + S_n' + T_1' + \cdots + T_k'.$$

It follows that $R_1 \times R_2 \subseteq S_1' + \cdots + S_n' + T_1' + \cdots T_k'$. Hence we conclude that $R_1 \times R_2 = S_1' + \cdots + S_n' + T_1' + \cdots + T_k'$ as left $R_1 \times R_2$-modules, which shows that $R_1 \times R_2$ is a left semisimple ring.

Problem 10. (After Chapter 19.3.) Let R be a ring.

(a) Show that R is a left semisimple ring if and only if R^{op} is a right semisimple ring.

(a) Use the Wedderburn–Artin theorem as well as Problems 2 and 5 to show that R is a left semisimple ring if and only if R is a right semisimple ring.

Solution.

(a) We only show that if R is a left semisimple ring, then R^{op} is a right semisimple ring, the converse is similar.

Assume that R is a left semisimple ring and we show that R^{op} is a right semisimple ring. Since R is left semisimple, we have that

$$R = S_1 + \cdots + S_n$$

for some simple left R-modules $S_1, \ldots, S_n$. Let $i \in \{1, \ldots, n\}$. By Remark 9.2(2), the left R-module S_i becomes a right R^{op}-module via defining

$$sr := rs$$

for every $s \in S_i$ and $r \in R$. Set $S = S_i$. We claim that $S_{R^{\mathrm{op}}}$ is a simple right R^{op}-module. Let $U \subseteq S_{R^{\mathrm{op}}}$ be a right R^{op}-submodule of $S_{R^{\mathrm{op}}}$. Then $ur \in U$ for every $u \in U$ and every $r \in r^{\mathrm{op}}$. Again, U becomes a left R-module via setting

$$ru := ur$$

for every $u \in U$ and $r \in R$. Then for every $u \in U$ and every $r \in R$ we have

$$ru = ur \in U$$

and clearly U is also closed under addition. Hence ${}_RU \subseteq {}_RS$ is a left R-submodule. Since ${}_RS$ is a simple R-module we obtain that ${}_RU = 0$ or ${}_RU = S$. Since as sets we have $U = {}_RU = U_{R^{\mathrm{op}}}$, we conclude that $U_{R^{\mathrm{op}}} = 0$ or $U_{R^{\mathrm{op}}} = S_{R^{\mathrm{op}}}$ and so $S_{R^{\mathrm{op}}}$ is a simple right R^{op}-module. Then, as sets, we have

$$R^{\mathrm{op}} = R = S_1 + \cdots + S_n,$$

showing that R^{op} is a semisimple right R^{op}-module and hence R^{op} is a right semisimple ring.

(b) We only show that if R is a left semisimple ring, then R^{op} is a left semisimple ring, the converse follows since $(R^{\mathrm{op}})^{\mathrm{op}} = R$.

Suppose that R is a left semisimple ring. By the Wedderburn–Artin theorem, we have that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

for some positive integers $n_1, \ldots, n_k$ and some division rings $D_1, \ldots, D_k$. Then

$$\begin{aligned} R^{\mathrm{op}} &\cong (M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k))^{\mathrm{op}} \\ &\cong (M_{n_1}(D_1))^{\mathrm{op}} \times \cdots \times (M_{n_k}(D_k))^{\mathrm{op}} && \text{(Problem 5(a))} \\ &\cong M_{n_1}(D_1^{\mathrm{op}}) \times \cdots \times M_{n_k}(D_k^{\mathrm{op}}). && \text{(Problem 5(b))} \end{aligned}$$

By Problem 2(a) we have that D_i^{op} is a division ring for all $i \in \{1, \ldots, k\}$. Set $D_i' := D_i^{\mathrm{op}}$. Hence we have shown that R^{op} is isomorphic to the ring

$$M_{n_1}(D_1') \times \cdots \times M_{n_k}(D_k'),$$

where $D_1', \ldots, D_k'$ are division rings, and so by the Wedderburn–Artin theorem, we have that R^{op} is a left semisimple ring.

Problem 11. (After Chapter 14.4.) (Exam December 2011, Problem 2.) Let F be a field and

$$R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} \middle| a, b, c, d, e \in F \right\}.$$

(a) Show that R is a subring of the full matrix ring $M_3(F)$.

(b) Show that both

$$I_1 = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are two-sided ideals of R.

(c) Give four equivalent conditions for a ring R to be *semisimple*. (For a definition of semisimple rings see Corollary 14.7 in the notes. You may use either results from the lectures, either results from other problems in this problem set freely.)

(d) Is R/I_1 a semisimple ring? Is R/I_2 a semisimple ring? Why or why not? *(Hint: find rings R_1 and R_2 such that $R/I_1 \cong R_1$ and $R/I_2 \cong R_2$.)*

Solution.

(a) Clearly $R \neq \varnothing$ since $0 \in R$. Let $A = \left(\begin{smallmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{smallmatrix}\right) \in R$. Then

$$A - B = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ 0 & d_1 - d_2 & 0 \\ 0 & 0 & e_1 - e_2 \end{pmatrix} \in R,$$

and

$$AB = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & 0 \\ 0 & 0 & e_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1d_2 & a_1c_2 + c_1e_2 \\ 0 & d_1d_2 & 0 \\ 0 & 0 & e_1e_2 \end{pmatrix} \in R,$$

and so R is a subring of $M_3(R)$.

(b) We first show that I_1 is a two-sided ideal of R. Let $X = \left(\begin{smallmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 0 & y_1 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I_1$ and $A = \left(\begin{smallmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{smallmatrix}\right)$. Then

$$X - Y = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & y_1 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1 - y_1 & 0 \\ 0 & x_2 - y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_1,$$

and

$$AX = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} \begin{pmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1x_1 + b_1x_2 & 0 \\ 0 & d_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_1,$$

and

$$XA = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} = \begin{pmatrix} 0 & x_1d_1 & 0 \\ 0 & x_2d_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_1,$$

and so I_1 is a two-sided ideal of R.

We now show that I_2 is a two-sided ideal of R. Let $X = \left(\begin{smallmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I_1$ and $A = \left(\begin{smallmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{smallmatrix}\right)$. Then

$$X - Y = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1 - y_1 & x_2 - y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_2,$$

and

$$AX = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1x_1 & a_1x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_2,$$

and

$$XA = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} = \begin{pmatrix} 0 & x_1d_1 & x_2e_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_2,$$

and so I_2 is a two-sided ideal of R.

(c) By Corollary 14.7 we have that R is semisimple if

(i) R is a left semisimple ring, i.e. ${}_RR$ is a semisimple module, or

(ii) R is a right semisimple ring, i.e. R_R is a right semisimple module.

By Problem 10 we have that these conditions are equivalent to

(iii) R^{op} being a left semisimple ring, or

(iv) R^{op} being a right semisimple ring.

Another condition is given by the Wedderburn-Artin theorem and is that

(v) there exist division rings $D_1, \ldots, D_k$ and positive integers $n_1, \ldots, n_k$ such that $R \cong M_{n_1}(D_1) \oplus \cdots M_{n_k}(D_k)$.

Finally, by Proposition 14.5 we have that another condition is given by

(vi) R is left artinian and has no nonzero nilpotent ideals.

(d) First let

$$R_1 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| a, b, c \in F \right\}.$$

Define a map $f_1 : R \to R_1$ by

$$f_1\left(\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}\right) = \begin{pmatrix} a & c \\ 0 & e \end{pmatrix}.$$

It is straightforward to check that f_1 is a ring homomorphism. Moreover, it is clear that f_1 is surjective. To compute the kernel of f_1 we have that

$$f_1\left(\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

implies

$$\begin{pmatrix} a & c \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so $a = c = e = 0$. Hence $\left(\begin{smallmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I_1$. Hence $\ker f_1 \subseteq I_1$. Clearly $I_1 \subseteq \ker f_1$ as well and so $\ker f_1 = I_1$. By the first isomorphism theorem for rings we obtain that

$$R/I_1 = R_1/\ker f_1 \cong \operatorname{Im} f_1 = R_1.$$

Hence we need to check if R_1 is semisimple. Consider the set

$$\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| x \in F \right\} \subseteq R_1.$$

It is easy to check that this is a two-sided ideal of R_1. On the other hand, it is nilpotent since $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all matrices $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$. Hence the ring is not semisimple by Proposition 14.5.

Now let

$$R_2 = F \oplus F \oplus F.$$

Define a map $f_2 : R \to R_2$ by

$$f_2\left(\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}\right) = (a, d, e).$$

It is straightforward to check that f_2 is a ring homomorphism. Moreover, it is clear that f_2 is surjective. To compute the kernel of f_2 we have that

$$f_2\left(\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}\right) = (0, 0, 0)$$

implies

$$(a, d, e) = (0, 0, 0)$$

and so $a = d = e = 0$. Hence $\left(\begin{smallmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I_2$. Hence $\ker f_2 \subseteq I_2$. Clearly $I_2 \subseteq \ker f_2$ as well and so $\ker f_2 = I_2$. By the first isomorphism theorem for rings we obtain that

$$R/I_2 = R_2/\ker f_2 \cong \operatorname{Im} f_2 = R_2 = F \oplus F \oplus F.$$

But $F \oplus F \oplus F$ is semisimple by the Wedderburn–Artin theorem. We conclude that R/I_2 is semisimple.

Problem 12. (After Chapter 19.2.) (Exercise 19.2.1 in the book.)

(a) Let R be a unital ring and let $M_1, \ldots, M_n$ be noetherian left R-modules. Show that their direct sum $M_1 \oplus \cdots \oplus M_n$ is a noetherian left R-module.

(b) Let $R_1, \ldots, R_n$ be a family of noetherian unital rings. Show that their direct product $R_1 \times \cdots \times R_n$ is again noetherian.

Solution.

(a) Clearly it is enough to show the claim for $n = 2$ since the general claim follows by induction. Let $f : M_1 \oplus M_2 \to M_1$ be the map given by $f((m_1, m_2)) = m_1$. It is easy to see that f is a homomorphism of left R-modules. Moreover, it is clearly surjective and so $\operatorname{Im} f = M_1$. Now we compute $\operatorname{Ker} f$. Assume that $f((m_1, m_2)) = 0$. Then $m_1 = 0$ and so

$$\ker f = \{(0, m_2) \mid m_2 \in M_2\} \cong M_2.$$

By the first isomorphism theorem for modules we obtain that

$$(M_1 \oplus M_2)/M_1 = (M_1 \oplus M_2)/\ker f \cong \operatorname{Im} f = M_2.$$

Since both M_1 and M_2 are noetherian by assumption, we conclude by Theorem 13.8 that $M_1 \oplus M_2$ is noetherian.

(b) We show that if $R_1, \ldots, R_n$ are left noetherian rings, then $R_1 \times \cdots \times R_n$ is a left noetherian ring. Similarly one can show that if $R_1, \ldots, R_n$ are right noetherian rings then $R_1 \times \cdots \times R_n$ is a right noetherian ring. Hence it follows that if $R_1, \ldots, R_n$ are noetherian (i.e. both left and right noetherian rings) then $R_1 \times \cdots \times R_n$ is a noetherian ring.

Clearly it is enough to show the claim for $n = 2$ since the general claim follows by induction. That is we want to show that $R_1 \times R_2$ is a noetherian left $R_1 \times R_2$-module. Let

$$R_1' = R_1 \times 0 = \{(r_1, 0) \mid r_1 \in R_1\}.$$

Then clearly R_1' is an $R_1 \times R_2$-submodule of $R_1 \times R_2$. We claim that R_1' is a noetherian left $R_1 \times R_2$-module. Let $U' \subseteq R_1'$ be an $R_1 \times R_2$-submodule of R_1'. Set

$$U = \{u \in R_1 \mid (u, 0) \in U'\}.$$

We claim that U is an R_1-submodule of ${}_{R_1}R_1$. Clearly $U \neq \varnothing$ since $U' \neq \varnothing$. Let $u, v \in U$ and $r \in R_1$. Then $(u, 0), (v, 0) \in U'$ and so $(u + v, 0) \in U'$ since U' is an $R_1 \times R_2$-submodule. This gives that $u + v \in U$, and so U is closed under addition. Moreover, we have that $(r, 0) \in R_1 \times R_2$ and so $(r, 0)(u, 0) = (ru, 0) \in U'$ sing U' is an $R_1 \times R_2$-submodule of R_1'. This gives that $ru \in U$ and so U is closed under scalar multiplication. Hence indeed U is an R_1-submodule of ${}_{R_1}R_1$. Since R_1 is left noetherian, we obtain that U is noetherian by Theorem 13.8(2). We conclude that U is finitely generated by Theorem 13.5(2). But then, if $\{u_1, \ldots, u_k\}$ is a set of R_1-generators of U, then $\{(u_1, 0), \ldots, (u_k, 0)\}$ is a set of $R_1 \oplus R_2$-generators of U' and so U' is finitely generated as an $R_1 \times R_2$-module. Since U' was arbitrary, we conclude that R_1' is indeed a noetherian $R_1 \oplus R_2$-module by Theorem 13.5(2).

Similarly we obtain that

$$R_2' = R_2 \times 0 = \{(0, r_2) \mid r_2 \in R_2\}$$

is a noetherian left $R_1 \times R_2$-module. Then we may consider the map $f : R_1 \times R_2 \to R_2$ given by $f((r_1, r_2)) = (0, r_2)$. This is clearly a homomorphism of left $R_1 \times R_2$ modules, with $\operatorname{Im} f = R_2$ and $\ker f = R_1$. Then by the first isomorphism theorem for modules we obtain that

$$(R_1 \times R_2)/R_1 = (R_1 \times R_2)/\ker f \cong \operatorname{Im} f = R_2.$$

Since both R_1 and $(R_1 \times R_2)/R_1$ are noetherian, we conclude that $R_1 \times R_2$ is also noetherian and so $R_1 \times R_2$ is a left noetherian ring.

Problem 13. (After Chapter 19.2.) Let R be a left noetherian unital ring.

(a) Let M be a finitely-generated left R-module. Show that M is a noetherian left R-module. *(You may want to use Problem 8(b) and Problem 12.)*

(b) (Exercise 19.2.6 in the book.) Show that the ring of $n \times n$ matrices $M_n(R)$ over R is also left noetherian. *(Hint: show that $M_n(R)$ is a noetherian left R-module using part (a). Then show that every left ideal of $M_n(R)$ is finitely generated as an R-module, and hence also as a left ideal of $M_n(R)$.)*

Solution.

(a) Since M is finitely generated, we have by Problem 8(b) that $M \cong R^k/K$ for some positive integer k and some submodule $K \subseteq R^k$. Since R, is noetherian as a left R-module, we have by Problem 12 that R^k is also noetherian as a left R-module. By Theorem 13.8(2) we have that R^k/K is also noetherian as a left R-module since R^k is clearly finitely generated. We conclude that $M \cong R^k/K$ is a noetherian left R-module.

(b) First note that $M_n(R)$ is a left R-module via the usual scalar multiplication of matrices. Clearly, it is also finitely generated as a left R-module with a generating set given by $\{e_{ij}\}_{1\leq i,j\leq n}$, where e_{ij} is the matrix with 1 in position ij and 0 everywhere else. Hence by part (a) it follows that $M_n(R)$ is a noetherian left R-module.

Now let $J \subseteq M_n(R)$ be a left ideal. Let I_n be the identity matrix. For $A \in J$ and $r \in R$ we set

$$rA \coloneqq (rI_n)A \in J,$$

It is straightforward to check that this makes J into a left R-module and hence to a submodule of $M_n(R)$. Since $M_n(R)$ is a noetherian left R-module, it follows that J is finitely generated as a left R-module by Theorem 13.5(2). Let $J = {}_R(A_1, \dots, A_k)$ for some $A_1, \dots, A_k$, that is the matrices $A_1, \dots, A_k$ generate J as a left R-module. Then for every $A \in J$ we have that there exist some $r_1, \dots, r_k \in R$ such that

$$A = r_1A_1 + \cdots + r_kA_k = (r_1I_n)A_1 + \cdots + (r_kI_n)A_k.$$

In particular, the matrices $\{A_1, \dots, A_k\}$ generate J as a left ideal of $M_n(K)$ as well. Hence J is finitely generated. Since J was arbitrary, we conclude that all left ideals of $M_n(R)$ are finitely generated and so $M_n(R)$ is a left noetherian ring by Corollary 13.6.

Problem 14. (After Chapter 19.2.) Let R be a unital ring. The aim of this problem is to show that R is left artinian and has no nonzero nilpotent ideals if and only if the left R-module ${}_RR$ is semisimple.

(a) Assume that R is left artinian. Let J

$$J = \sum_{I \subseteq R \text{ minimal ideal}} I,$$

that is J is the sum of all minimal ideals of R. Show that J is the sum of a finite family of minimal ideals of R. Show that if moreover R has no nonzero nilpotent ideals, then the left R-module ${}_RR$ is semisimple. *(Hint: for the second claim apply Lemma 14.4 on J and then use Problem 5 in Problem Set 2.)*

(b) Assume that the left R-module ${}_RR$ is semisimple. Show that ${}_RR$ is the direct sum of a finite family of simple left R-modules. Conclude that R is left artinian. *(Hint: use Theorem 11.5(2) and consider the element $1 \in R$.)* Show also that there exists no nonzero nilpotent ideal of R. *(Hint: use Theorem 11.5(3) and consider the element $1 \in R$.)*

Solution.

(a) Assume first that R is left artinian. Let I be any left ideal of R. Let

$$\mathcal{S}(I) = \{L \subseteq I \mid L \text{ is a left ideal and } L \neq 0\}.$$

If $I \neq 0$, then $I \in \mathcal{S}(I)$ and so $\mathcal{S}(I) \neq \varnothing$. Since R is left artinian, in this case $\mathcal{S}(I)$ has a minimal element by Corollary 13.6. In particular, this minimal element is also a minimal ideal by definition. Hence, we have that $\mathcal{S}(R)$ has a minimal element, which means that there exists a minimal ideal in R. Hence J is a nonempty sum. We claim that there exists a finite set of minimal left ideals $\{I_1, \dots, I_k\}$ such that $J = \sum_{i=1}^k I_i$. Indeed, if this is not the case, then there exist infinitely many minimal left ideals $I_1, I_2, \dots, I_k, \dots$ and a descending chain of inclusions

$$J \supseteq \sum_{i=1}^{\infty} I_i \supsetneq \sum_{i=2}^{\infty} I_i \supsetneq \sum_{i=3}^{\infty} I_i \supsetneq \cdots,$$

which contradicts that R is left artinian. Hence $R = \sum_{i=1}^{k} I_i$ for some minimal left ideals I_i.

Now assume moreover that there exist no nonzero nilpotent ideals of R. By Lemma 14.4, we have that $J = Re$ for some idempotent $e \in R$. Then by Problem 5 in Problem Set 2 (with right ideals replaced by left ideals) we obtain that $R = Re \oplus R(1-e) = J \oplus R(1-e)$. Assume towards a contradiction that $R(1-e) \neq 0$. Then $\mathcal{S}(R(1-e))$ has a minimal element say K. Then $K \subseteq R(1-e)$ and K is a minimal ideal. In particular, K appears as a summand of J and so $J \cap K \neq 0$. But then

$$0 = J \cap R(1-e) \supseteq J \cap K \neq 0,$$

which is a contradiction. Hence $R(1-e) = 0$ and so $R = Re = J$. But then R is the sum of simple submodules of R and so it is semisimple by Theorem 11.5(1).

(b) Now assume that ${}_RR$ is a semisimple left R-module. Then by Theorem 11.5(2) we have that ${}_RR = \bigoplus_{\lambda \in \Lambda} S_\lambda$ where $\{S_\lambda\}_{\lambda \in \Lambda}$ is a family of simple left R-modules. Since $1 \in_R R = \bigoplus_{\lambda \in \Lambda} S_\lambda$, we have that there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that

$$1 = \sum_{\lambda \in \Lambda_0} s_\lambda,$$

where $s_\lambda \in S_\lambda$. Then for every $r \in R$ we have

$$r = r1 = r \sum_{\lambda \in \Lambda_0} s_\lambda = \sum_{\lambda \in \Lambda_0} r s_\lambda \in \bigoplus_{\lambda \in \Lambda_0} S_\lambda,$$

and so $R \subseteq \bigoplus_{\lambda \in \Lambda_0} S_\lambda$. Since the opposite set inclusion clearly holds, we conclude that $R = \bigoplus_{\lambda \in \Lambda_0} S_\lambda$. Since Λ_0 is a finite set, we may relabel the modules S_λ for $\lambda \in \Lambda_0$ to obtain that

$$R = \bigoplus_{i=1}^{n} S_i.$$

For every $i \in \{1, \dots, n\}$ we have that S_i is a simple left R-module, and so $S_i = (a_i)$ for some $a_i \in S_i \setminus \{0\}$. Now let

$${}_RR = \bigoplus_{i=1}^{n} S_i \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

be an infinite descending chain of submodules. Notice that if $a_i \in M_k$, then $S_i = (a_i) \subseteq M_k$. Hence in the above, any proper inclusion $M_k \supsetneq M_{k+1}$ indicates that there exists $i \in \{1, \dots, n\}$ such that $a_i \in M_k$ but $a_i \notin M_{k+1}$. We conclude that there can be at most n proper inclusions in such a chain of submodules and so R is left artinian.

Now let $I \subseteq R$ be a left ideal of R with $I \neq 0$ and it remains to show that I is not nilpotent. Since ${}_RR$ is semisimple, and $I \subseteq {}_RR$ is a submodule, by Theorem 11.5(3) there exists a submodule $L \subseteq {}_RR$ such that $I \oplus L = {}_RR$. Then $1 \in {}_RR = I \oplus L$ implies that $1 = e + f$ for some $e \in I$ and $f \in L$. Clearly, e cannot be 0 since otherwise for every $a \in I$ we have

$$a = a1 = a(e+f) = a(0+f) = af \in N$$

and so $a \in I \cap N$ implies $a = 0$, which contradicts $I \neq 0$. Multiplying $1 = e + f$ with e from the left, we obtain that

$$e = e^2 + ef$$

or $e - e^2 = ef$. Then $ef \in N$ since $e \in R$ and N is a left R-module, while $e \in I$ and $e^2 \in I$ imply that $ef = e - e^2 \in I$. Hence $ef \in I \cap N = \{0\}$ and so $ef = 0$. We conclude that $e^2 = e$ and so e is idempotent. But then $0 \neq e = e^n \in I^n$ for any $n > 0$ and so I is not nilpotent.

Problem 15. (After Chapter 19.3.) Let R be a unital ring.

(a) Show that R is left noetherian if and only if R^{op} is right noetherian. *(Hint: use Corollary 13.6.)*

(b) Show that R is left artinian if and only if R^{op} is right artinian. *(Hint: use Corollary 13.6.)*

(c) Show that a semisimple ring is both left and right artinian and noetherian (you may use Example 13.7(5) and Problems 10(a) and 14).

Solution.

(a) We only show that if R is left noetherian then R^{op} is right noetherian since the converse is similar.

Assume that R is left noetherian. Let I be a right ideal of R^{op}. Then I is a left ideal of R. Since R is left noetherian, it follows by Corollary 13.6(2) that I is finitely generated as a left ideal of R. It readily follows that it is finitely generated as a right ideal of R^{op}. Hence by Corollary 13.6(2) (where we replace "left" by "right") we obtain that R^{op} is right noetherian.

(b) We only show that if R is left artinian then R^{op} is right artinian since the converse is similar.

Assume that R is left artinian. Let S be a set of right ideals of R^{op}. Since every right ideal of R^{op} is a left ideal of R, it follows that S is a set of left ideals of R. It follows that S is a set of left ideals of R, and so by Corollary 13.6(3) it has a minimal element. In particular this minimal element of S is a left ideal of R and so a right ideal of R^{op}. Hence by Corollary 13.6(3) (where we replace "left" by "right") we conclude that R^{op} is right artinian.

(c) Let R be a semisimple ring. By Corollary 14.7 this means that R is both left semisimple and right semisimple. By Problem 14 we have that since R is a left semisimple ring, it is left artinian. By Problem 10(a) we have that since R is a right semisimple ring, R^{op} is a left semisimple ring. It follows by Problem 14 that R^{op} is left artinian and so by part (b) it follows that R is right artinian. Since R is a unital ring which is both left and right artinian, it follows that it is also left and right noetherian by Example 13.7(5).

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 16. (After Chapter 19.2.) Let R be the ring

$$R = \left\{ \begin{pmatrix} p & q \\ 0 & n \end{pmatrix} \mid p, q \in \mathbb{Q}, n \in \mathbb{Z} \right\},$$

abbreviated as $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$.

(a) Find all left ideals in R.

(b) Show that any left ideal can be generated by at most 2 elements.

(c) Is R left noetherian?

(d) Find all right ideals in R. Are they all finitely generated?

(e) Is R right noetherian?

Solution.

(a) Let I be a left ideal in R and assume that $I \neq (0)$. Let $\left(\begin{smallmatrix} p & q \\ 0 & n \end{smallmatrix}\right) \in I$. For any $k \in \mathbb{Z}$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} p & q \\ 0 & n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & kn \end{pmatrix},$$

hence $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (n) \end{smallmatrix}\right) \subseteq I$. We consider several cases.

- Assume that there exists such a matrix with $n \neq 0$ and let n be minimal such that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (n) \end{smallmatrix}\right) \subseteq I$ and $n > 0$. Assume to a contradiction that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (m) \end{smallmatrix}\right) \subseteq I$ for some $m \notin (n)$. Then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & kn+lm \end{smallmatrix}\right) \in I$ for all

$k, l \in \mathbb{Z}$ and so by Bezout's theorem we obtain that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I$. By minimality of n we conclude that $n = 1$, contradicting $m \notin (n) = \mathbb{Z}$. Therefore, we conclude that in this case $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (n) \end{smallmatrix}\right) \subseteq I$ for some minimal $n \geq 1$, or that $(\left(\begin{smallmatrix} 0 & 0 \\ 0 & n \end{smallmatrix}\right)) \subseteq I$. Moreover, for any $x \in \mathbb{Q}$ we have that

$$\begin{pmatrix} 0 & \frac{x}{n} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & n \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix},$$

hence $\left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right) \subseteq (\left(\begin{smallmatrix} 0 & 0 \\ 0 & n \end{smallmatrix}\right))$. In other words, we have shown that

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}\right) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{pmatrix} = \left\{ \begin{pmatrix} 0 & q \\ 0 & kn \end{pmatrix} \,\middle|\, q \in \mathbb{Q}, k \in \mathbb{Q} \right\}.$$

Now assume that $p = 0$. Then for any $\left(\begin{smallmatrix} p' & q' \\ 0 & n' \end{smallmatrix}\right) \in R$ we have

$$\begin{pmatrix} p' & q' \\ 0 & n' \end{pmatrix} \begin{pmatrix} 0 & q \\ 0 & n \end{pmatrix} = \begin{pmatrix} 0 & p'q' + q'n \\ 0 & n'n \end{pmatrix},$$

and so in this case we have that

$$I = \left(\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}\right) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{pmatrix}.$$

Finally assume that $p \neq 0$. Then $\left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{smallmatrix}\right) \subseteq I$ implies that $\left(\begin{smallmatrix} 0 & -q \\ 0 & -n \end{smallmatrix}\right) \in I$ and so

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & q \\ 0 & n \end{pmatrix} + \begin{pmatrix} 0 & -q \\ 0 & -n \end{pmatrix} \in I.$$

Then for any $x \in \mathbb{Q}$ we have that

$$\begin{pmatrix} \frac{x}{p} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in I$$

and so $\left(\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subseteq I$. Therefore, in this case we obtain that

$$I = \left(\begin{pmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{pmatrix}\right) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & (n) \end{pmatrix}.$$

• Assume now that there exists no matrix with $n \neq 0$ in I. Then $I \subseteq \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$. Let $\left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$ and let $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in R$. We have that

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xp & xq \\ 0 & 0 \end{pmatrix},$$

and so we may view $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$ as a $\mathbb{Q}$-module isomorphic to $\mathbb{Q}^2$. Then I is a left ideal of $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$, and being closed under multiplication on the left with elements of $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$ coincides with being closed under scalar multiplication in the $\mathbb{Q}$-module $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$. Hence I can be identified with a subspace of $\mathbb{Q}^2$. Therefore, in this case we have that

$$I = (0) \text{ or } I = \left(\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} xp & xq \\ 0 & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{Q} \right\} \text{ or } I = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}.$$

All in all, we conclude that the left ideals of R are given from the following list

$$0, \; \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{pmatrix}, \; \left(\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} xp & xq \\ 0 & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{Q} \right\}, \; \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & (n) \end{pmatrix}, \; R.$$

(b) From part (a) we have a list of the ideals. Clearly 0 is finitely generated and R is finitely generated since R is unital. Also $(\left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right))$ is finitely generated by construction, and $(\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))$. Furthermore, we have shown in part (a) that $(\left(\begin{smallmatrix} 0 & 0 \\ 0 & n \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & (n) \end{smallmatrix}\right)$ is finitely generated, and so $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & (n) \end{smallmatrix}\right) = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & (n) \end{smallmatrix}\right))$ is finitely generated as well. Hence all left ideals in R are generated by at most two elements.

(c) By Corollary 13.6 it follows that R is left noetherian.

(d) Let I be a right ideal in R and let $\left(\begin{smallmatrix} p & q \\ 0 & n \end{smallmatrix}\right) \in I$. For any $x \in \mathbb{Q}$ we have that

$$\begin{pmatrix} p & q \\ 0 & n \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} px & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence if $p \neq 0$, we conclude that $\left(\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subseteq I$.

- Assume that $p = 0$ and $q = 0$. Then as in the previous case it is easy to see that

$$I = \begin{pmatrix} 0 & 0 \\ 0 & (n) \end{pmatrix}.$$

- Now assume that $p = 0$ and $n = 0$. For any $k \in \mathbb{Z}$ we have

$$\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} 0 & kq \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\left(\begin{pmatrix} 0 & q_1 \\ 0 & 0 \end{pmatrix}\right) \subseteq \left(\begin{pmatrix} 0 & q_2 \\ 0 & 0 \end{pmatrix}\right) \text{ if and only if } q_1 = mq_2 \text{ for some } m \in \mathbb{Z}.$$

More generally, let J be an index set (not neccesarily finite) and let $q \in \mathbb{Q}$ and $q_j \in \mathbb{Q}$ for $j \in J$. Then

$$\left(\left\{\begin{pmatrix} 0 & q_j \\ 0 & 0 \end{pmatrix} \,\middle|\, j \in J\right\}\right) \subseteq \left(\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}\right) \text{ if and only if } q_j = m_j q \text{ for some } m_j \in \mathbb{Z}.$$

We conclude that for any sequence $q_1, q_2, \ldots$ of rational numbers such that for each q_j there exists $j' < j$ with $q_{j'} \notin \mathbb{Z}q_j$, we obtain an ideal

$$I(q_j \mid j \in J) = \begin{pmatrix} 0 & \sum_{j \in J} \mathbb{Z}q_j \\ 0 & 0 \end{pmatrix}$$

as above, and $I = I(q_j \mid j \in J)$.

- Now assume that $p = 0$ but both $q \neq 0$ and $n \neq 0$ hold. Then

$$\left(\begin{pmatrix} 0 & q \\ 0 & n \end{pmatrix}\right) = \left\{\begin{pmatrix} 0 & kq \\ 0 & kn \end{pmatrix} \,\middle|\, k \in \mathbb{Z}\right\} = \mathbb{Z}\begin{pmatrix} 0 & q \\ 0 & n \end{pmatrix}.$$

Similarly as to the previous case, we have that for any sequence $(q_1, n_1), (q_2, n_2), \ldots \in \mathbb{Q} \times \mathbb{Z}$ we have that

$$\left(\left\{\begin{pmatrix} 0 & q_j \\ 0 & n_j \end{pmatrix} \,\middle|\, j \in J\right\}\right) \subseteq \left(\begin{pmatrix} 0 & q \\ 0 & n \end{pmatrix}\right) \text{ if and only if } q_j = m_j q \text{ and } n_j = m_j n \text{ for some } m_j \in \mathbb{Z}.$$

Therefore, for any sequence $(q_1, n_1), (q_2, n_2), \ldots \in \mathbb{Q} \times \mathbb{Z}$ such that for each (q_j, n_j) there exists $j' < j$ such that for all $m_j \in \mathbb{Z}$ we have $q_{j'} \neq m_j q_j$ or $n_{j'} \neq m_j n_j$, we obtain an ideal

$$I((q_j, n_j) \mid j \in J) = \sum_{j \in J} \mathbb{Z}\begin{pmatrix} 0 & q_j \\ 0 & n_j \end{pmatrix}$$

as above, and $I = I((q_j, n_j) \mid j \in J)$.

Considering all of the above, we conclude that the right ideals of R are given from the following list

$$0, \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & (n) \end{pmatrix}, \sum_{j \in J} \mathbb{Z}\begin{pmatrix} 0 & q_j \\ 0 & 0 \end{pmatrix}, \sum_{j \in J} \mathbb{Z}\begin{pmatrix} 0 & q_j \\ 0 & n_j \end{pmatrix}, \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & (n) \end{pmatrix}, \sum_{j \in J} \mathbb{Z}\begin{pmatrix} \mathbb{Q} & q_j \\ 0 & 0 \end{pmatrix}, \sum_{j \in J} \mathbb{Z}\begin{pmatrix} \mathbb{Q} & q_j \\ 0 & n_j \end{pmatrix}, R.$$

These are not all finitely generated. For example assume to a contradiction that there exist $A_1, \ldots, A_k \in R$ such that

$$I(1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots) = \sum_{j=1}^{\infty} \mathbb{Z} \begin{pmatrix} 0 & \frac{1}{2^j} \\ 0 & 0 \end{pmatrix} = (A_1, \ldots, A_k).$$

Using the list above we see that we may assume that $A_i = \left(\begin{smallmatrix} 0 & q_i \\ 0 & 0 \end{smallmatrix}\right)$. Write $q_i = \frac{a_i}{b_i}$ with $a_i, b_i \in \mathbb{Z}$ and set $q = \frac{a_1 \cdot \ldots \cdot a_k}{b_1 \cdot \ldots \cdot b_k}$. Then

$$q_i = m_i q, \text{ for } m_i = \tfrac{b_1 \cdot \ldots \cdot b_k}{b_i} \in \mathbb{Z},$$

and so

$$\sum_{j=1}^{\infty} \mathbb{Z} \begin{pmatrix} 0 & \frac{1}{2^j} \\ 0 & 0 \end{pmatrix} \subseteq (A_1, \ldots, A_k) \subseteq \left(\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \right).$$

Hence there exist $m_j \in \mathbb{Z}$ such that $\frac{1}{2^j} = m_j q$ for all $j \geq 0$. Then for $j = 0$ we obtain that $1 = m_0 q$ and so $q = 1$ or $q = -1$ since $m_0 \in \mathbb{Z}$. But then $\frac{1}{2} = m_1 q$ is a contradiction. Therefore, $I(1, \frac{1}{2}, \frac{1}{4}, \ldots)$ is not finitely generated.

(e) In part (d) we showed that there exist ideals in R which are not finitely generated. By Corollary 13.6 we conclude that R is not right noetherian.

Problem 17. (After Chapter 19.3.) Let R be a finite ring such that $r^2 = r$ for all $r \in R$.

(a) Show that R is commutative. *(Hint: consider the elements $(r+s)^2$ and $r^2 = (-r)^2$ for $r, s \in R$.)*

(b) Show that R is a left artinian ring with no nontrivial nilpotent ideals.

(c) Use the Wedderburn–Artin theorem to show that R is isomorphic as a ring to $\mathbb{Z}_2^n = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}}$ for some $n \geq 1$.

Solution.

(a) For every $r \in R$ we have

$$r = r^2 = (-r)^2 = -r. \tag{1}$$

Moreover, for any $r, s \in R$ we have

$$r + s = (r+s)^2 = (r+s)(r+s) = r^2 + rs + sr + s^2 = r + rs + sr + s,$$

and so we obtain that $rs + sr = 0$ or $rs = -(sr)$. Then, using (1) applied to $sr \in R$ we obtain

$$rs = -sr = sr,$$

and so R is commutative.

(b) Let I be a nilpotent ideal of R and let $x \in I$. Then $I^n = 0$ for some $n \geq 1$. Since $x^n \in I^n$, we obtain that $x^n = 0$. Then using $x = x^2$ we obtain

$$x = x^2 = x^3 = \cdots = x^n = 0,$$

and so $x = 0$. Hence $I = 0$ is the zero ideal. This shows that there are no nontrivial nilpotent ideals in R.

Now let

$$R \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

be a sequence of left R-submodules of R. Since R is finite, we have that there are finitely many strict inclusions in the above sequence. We conclude that the above sequence stabilizes and so R is a left artinian ring.

(c) By Proposition 14.5 we have that R is a left semisimple ring. By the Wedderburn–Artin theorem we obtain that there exist division rings $D_1, \dots, D_k$ and positive integers $n_1, \dots, n_k$ such that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Let $1 \leq p \leq k$ and let $e_{ij} \in M_{n_p}(D_p)$ be the matrix with 1 in position (i, j) and 0 everywhere else. Then

$$(0, \dots, 0, e_{ij}, 0, \dots, 0) = (0, \dots, 0, e_{ij}, 0, \dots, 0)^2 = (0, \dots, 0, e_{ij}^2, 0, \dots, 0)$$

gives $e_{ij} = e_{ij}^2$. If $i \neq j$, then

$$0 \neq e_{ij} = e_{ij}^2 = 0,$$

where the first inequality follows since $1 \neq 0$ in a division ring. It follows that $i = j$. Hence we conclude that $n_p = 1$. Since p was arbitrary, we conclude that $n_1 = \cdots = n_k = 1$ and so

$$R \cong D_1 \times \cdots \times D_k.$$

Now let $1 \leq p \leq k$ and $x \in D_p$. Then

$$(0, \dots, 0, x, 0, \dots, 0) = (0, \dots, 0, x, 0, \dots, 0)^2 = (0, \dots, 0, x^2, 0, \dots, 0)$$

gives $x = x^2$. We obtain that

$$x(x-1) = 0$$

in D_p. Since D_p is a division ring, it is also an integral domain and so $x = 1$ or $x = 0$ in D_p. Therefore $D_p = \{0, 1\} \cong \mathbb{Z}_2$ since this is the only ring with two elements. We conclude that

$$R \cong \mathbb{Z}_2^k = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{k \text{ copies}}$$

as required.