Rings and modules - Problem set 4

To be solved on Tuesday 31.10

Problem 1. (After Chapter 14.5.) Show that $\{2,3\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module, but no subset of $\{2,3\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}$.

Problem 2. (After Chapter 14.4.) Let D be a division ring and $n > 0$ a positive integer. Let $R = M_n(D)$ and let $e_{kk} \in R$ be the matrix with 1 in position (k,k) and 0 everywhere else.

(a) Show that D^{op} is a division ring.

(b) Show that ${}_RR = Re_{11} \oplus \cdots \oplus Re_{kk}$ and so R is a left semisimple ring.

Problem 3. (After Chapter 19.2.) (Exercise 19.2.3 in the book.) Show that every principal left ideal unital ring is left noetherian.

Problem 4. (After Chapter 14.3.) Let R be a ring that is not unital. Let $\tilde{R} = R \times \mathbb{Z}$. For $(r,n),(s,k) \in \tilde{R}$ define

$$(r,n) + (s,k) \coloneqq (r+s, n+k), \quad (r,n)(s,k) \coloneqq (rs + kr + ns, nk).$$

(a) Show that $\tilde{R}$ is a unital ring with $1_{\tilde{R}} = (0,1)$.

(b) Let M be a left R-module. For $m \in M$ and for $(r,n) \in \tilde{R}$ define

$$(r,n)m \coloneqq rm + nm.$$

Show that this makes M a left $\tilde{R}$-module. Also show that if $f : {}_RM \to {}_RN$ is an homomorphism of left R-modules, then $f : {}_{\tilde{R}}M \to {}_{\tilde{R}}N$ is also a homomorphism of left $\tilde{R}$-modules.

(c) Let $\tilde{M}$ be a left $\tilde{R}$-module. For $\tilde{m} \in \tilde{M}$ and for $r \in R$ define

$$r\tilde{m} \coloneqq (r,0)\tilde{m}.$$

Show that this makes $\tilde{M}$ a left R-module. Also show that if $\tilde{f} : {}_{\tilde{R}}\tilde{M} \to {}_{\tilde{R}}\tilde{N}$ is a homomorphism of left $\tilde{R}$-modules, then $\tilde{f} : {}_RM \to {}_RN$ is a homomorphism of left R-modules.

(d) Show that the two constructions in (b) and (c) are inverse to each other.

Thus this problem shows that studying left R-modules and homomorphisms of left R-modules is the same as studying $\tilde{R}$-modules and homomorphisms of left $\tilde{R}$-modules, motivating the fact that we focus on unital rings.

Problem 5. (After Chapter 10.2.) Let R and S be rings and $n \geq 1$ a positive integer.

(a) Show that the rings $(R \times S)^{\mathrm{op}}$ and $R^{\mathrm{op}} \times S^{\mathrm{op}}$ are isomorphicm.

(b) Show that the rings $(M_n(R))^{\mathrm{op}}$ and $M_n(R^{\mathrm{op}})$ are isomorphic.

Problem 6. (After Chapter 19.3.) (Exercise 19.3.1 in the book.) Let R be a left artinian unital ring with no nonzero nilpotent ideals. Show that for each two-sided ideal I of R, R/I is also left artinian with no nonzero nilpotent ideals. *(Hint: use Proposition 14.5 and Wedderburn–Artin theorem.)*

Problem 7. (After Chapter 10.3.) (Exam December 2009, Problem 4.) Let A be a left ideal in a unital ring R and assume that $A = Aa$ for some $a \neq 0$ in A.

(a) Show that there is some $e \in A$ where $ea \neq 0$ and $(e^2 - e)a = 0$.

(b) Let $B = \{x \in A \mid xa = 0\}$. Show that B is a left ideal in R.

(c) Assume further that the left ideal A is a minimal left ideal. Show that the element e from (a) is then an idempotent element.

(You can use (a) to show (b) even if you do not show (a), and you can use (a) and (b) to show (c).)

Problem 8. (After Chapter 14.5.) Let R be a unital ring. The aim of this problem is to show that every left R-module is the quotient of a free R-module.

(a) Let I be a set and let

$$R^I = \bigoplus_{i \in I} R = \{(r_i)_{i \in I} \mid r_i \in R \text{ and finitely many } r_i \neq 0\}.$$

Show that R^I is a free left R-module with a basis indexed by I.

(b) Let M be a left R-module. Show that there is a free R-module F and a submodule $K \subseteq F$ such that $M \cong F/K$. Show that if M is finitely generated, then F can also be taken to be finitely generated. *(Hint: use part (a) with $I = M$.)*

Problem 9. (After Chapter 19.3.) Let $R_1, \ldots, R_m$ be unital left semisimple rings. Show that $R_1 \times \cdots \times R_m$ is a left semisimple ring, without using the Wedderburn–Artin theorem.

Problem 10. (After Chapter 19.3.) Let R be a ring.

(a) Show that R is a left semisimple ring if and only if R^{op} is a right semisimple ring.

(a) Use the Wedderburn–Artin theorem as well as Problems 2 and 5 to show that R is a left semisimple ring if and only if R is a right semisimple ring.

Problem 11. (After Chapter 14.4.) (Exam December 2011, Problem 2.) Let F be a field and

$$R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} \middle| a, b, c, d, e \in F \right\}.$$

(a) Show that R is a subring of the full matrix ring $M_3(F)$.

(b) Show that both

$$I_1 = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are two-sided ideals of R.

(c) Give four equivalent conditions for a ring R to be *semisimple*. (For a definition of semisimple rings see Corollary 14.7 in the notes. You may use either results from the lectures, either results from other problems in this problem set freely.)

(d) Is R/I_1 a semisimple ring? Is R/I_2 a semisimple ring? Why or why not? *(Hint: find rings R_1 and R_2 such that $R/I_1 \cong R_1$ and $R/I_2 \cong R_2$.)*

Problem 12. (After Chapter 19.2.) (Exercise 19.2.1 in the book.)

(a) Let R be a unital ring and let $M_1, \ldots, M_n$ be noetherian left R-modules. Show that their direct sum $M_1 \oplus \cdots \oplus M_n$ is a noetherian left R-module.

(b) Let $R_1, \ldots, R_n$ be a family of noetherian unital rings. Show that their direct product $R_1 \times \cdots \times R_n$ is again noetherian.

Problem 13. (After Chapter 19.2.) Let R be a left noetherian unital ring.

(a) Let M be a finitely-generated left R-module. Show that M is a noetherian left R-module. *(You may want to use Problem 8(b) and Problem 12.)*

(b) (Exercise 19.2.6 in the book.) Show that the ring of $n \times n$ matrices $M_n(R)$ over R is also left noetherian. *(Hint: show that $M_n(R)$ is a noetherian left R-module using part (a). Then show that every left ideal of $M_n(R)$ is finitely generated as an R-module, and hence also as a left ideal of $M_n(R)$.)*

Problem 14. (After Chapter 19.2.) Let R be a unital ring. The aim of this problem is to show that R is left artinian and has no nonzero nilpotent ideals if and only if the left R-module ${}_RR$ is semisimple.

(a) Assume that R is left artinian. Let J

$$J = \sum_{I \subseteq R \text{ minimal ideal}} I,$$

that is J is the sum of all minimal ideals of R. Show that J is the sum of a finite family of minimal ideals of R. Show that if moreover R has no nonzero nilpotent ideals, then the left R-module ${}_RR$ is semisimple. *(Hint: for the second claim apply Lemma 14.4 on J and then use Problem 5 in Problem Set 2.)*

(b) Assume that the left R-module ${}_RR$ is semisimple. Show that ${}_RR$ is the direct sum of a finite family of simple left R-modules. Conclude that R is left artinian. *(Hint: use Theorem 11.5(2) and consider the element $1 \in R$.)* Show also that there exists no nonzero nilpotent ideal of R. *(Hint: use Theorem 11.5(3) and consider the element $1 \in R$.)*

Problem 15. (After Chapter 19.3.) Let R be a unital ring.

(a) Show that R is left noetherian if and only if R^{op} is right noetherian. *(Hint: use Corollary 13.6.)*

(b) Show that R is left artinian if and only if R^{op} is right artinian. *(Hint: use Corollary 13.6.)*

(c) Show that a semisimple ring is both left and right artinian and noetherian (you may use Example 13.7(5) and Problems 10(a) and 14).

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 16. (After Chapter 19.2.) Let R be the ring

$$R = \left\{ \begin{pmatrix} p & q \\ 0 & n \end{pmatrix} \mid p, q \in \mathbb{Q}, n \in \mathbb{Z} \right\},$$

abbreviated as $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$.

(a) Find all left ideals in R.

(b) Show that any left ideal can be generated by at most 2 elements.

(c) Is R left noetherian?

(d) Find all right ideals in R. Are they all finitely generated?

(e) Is R right noetherian?

Problem 17. (After Chapter 19.3.) Let R be a finite ring such that $r^2 = r$ for all $r \in R$.

(a) Show that R is commutative. *(Hint: consider the elements $(r+s)^2$ and $r^2 = (-r)^2$ for $r, s \in R$.)*

(b) Show that R is a left artinian ring with no nontrivial nilpotent ideals.

(c) Use the Wedderburn–Artin theorem to show that R is isomorphic as a ring to $\mathbb{Z}_2^n = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}}$ for some $n \geq 1$.