Rings and modules - Problem set 3 solutions

Solved on Tuesday 17.10

Problem 1. (After Chapter 14.3.) Let R be a ring and let $f : M \to N$ be a homomorphism of left R-modules. Show that the following hold.

(a) $f(0_M) = 0_N$.

(b) $f(m_1 - m_2) = f(m_1) - f(m_2)$ for all $m_1, m_2 \in M$.

(c) The set $\operatorname{Im} f = \{f(m) \mid m \in M\}$ is a submodule of N.

(d) The set $\operatorname{Ker} f = \{m \in M \mid f(m) = 0_N\}$ is a submodule of M.

(e) f is injective if and only if $\operatorname{Ker} f = \{0_M\}$.

(f) f is surjective if and only if $\operatorname{Im} f = N$.

Solution.

(a) We have that
$$f(0_M) = f(0_M + 0_M) = f(0_M) + f(0_M)$$
and so $f(0_M) = 0_N$.

(b) For all $m \in M$ we have
$$0_N = f(0_M) = f(m - m) = f(m) + f(-m)$$
and so $f(-m) = -f(m)$. Hence for all $m_1, m_2 \in M$ we obtain $f(m_1 - m_2) = f(m_1) + f(-m_2) = f(m_1) - f(m_2)$.

(c) By part (a) we have that $0_N \in \operatorname{Im} f$ and so $\operatorname{Im} f \neq \varnothing$. Let $n_1, n_2 \in \operatorname{Im} f$ and $r \in R$. Then there exist $m_1, m_2 \in M$ such that $f(m_1) = n_1$ and $f(m_2) = n_2$. Using part (b) we obtain
$$f(m_1 - m_2) = f(m_1) - f(m_2) = n_1 - n_2 \in \operatorname{Im} f.$$
Moreover, we also have
$$f(rm_1) = rf(m_1) = rn_1 \in \operatorname{Im} f,$$
which shows that $\operatorname{Im} f$ is a submodule of N.

(d) By part (a) we have that $0_M \in \operatorname{Ker} f$ and so $\operatorname{Ker} f \neq \varnothing$. Let $m_1, m_2 \in \operatorname{Ker} f$ and $r \in R$. Then $f(m_1) = 0_N$ and $f(m_2) = 0_N$. Using part (b) we obtain
$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0_N - 0_N = 0_N,$$
and so $m_1 - m_2 \in \operatorname{Ker} f$. Moreover we also have
$$f(rm_1) = rf(m_1) = r0_N = 0_N,$$
and so $rm_1 \in \operatorname{Ker} f$, which shows that $\operatorname{Ker} f$ is a submodule of M.

(e) Assume that f is injective and let $m \in \operatorname{Ker} f$. Then using part (a) we have that $f(m) = 0_N = f(0_M)$ and so injectivity of f gives $m = 0_M$. Therefore $\operatorname{Ker} f \subseteq \{0_M\}$ from which we conclude that $\operatorname{Ker} f = \{0_M\}$.

Assume now that $\operatorname{Ker} f = \{0_M\}$. Let $m_1, m_2 \in M$ be such that $f(m_1) = f(m_2)$. Then using part (b) we have

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0_N,$$

and so $m_1 - m_2 \in \operatorname{Ker} f = \{0_M\}$. Hence $m_1 - m_2 = 0_M$ from which we obtain that $m_1 = m_2$ and so f is injective.

(f) This follows immediately by the definition of $\Im f$.

Problem 2. (After Chapter 14.4.) Let F be a field and $R = M_n(F)$. Let $k \in \{1, \dots, n\}$ and e_{kk} be the matrix with 1 in position (k, k) and zero everywhere else. Show that

$$Re_{kk} = \left\{ \begin{pmatrix} 0 & \cdots & 0 & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & 0 & \cdots & 0 \end{pmatrix} \middle| a_{ik} \in F \right\}$$

is a simple left R-module.

Solution. We have that $Re_{kk} = (e_{kk})$ is a left R-module. To show that it is simple, by Theorem 11.5 it is enough to show that $Re_{kk} = (A)$ for any $A \in Re_{kk} \setminus \{0\}$. Let

$$A = \begin{pmatrix} 0 & \cdots & 0 & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & 0 & \cdots & 0 \end{pmatrix} \in Re_{kk} \setminus \{0\}.$$

Then $a_{jk} \neq 0$ for some $j \in \{1, \dots, n\}$. We have that $a_{jk}^{-1} e_{kj} \in R$ and so a direct computation shows that

$$a_{jk}^{-1} e_{kj} A = e_{kk} \in (A).$$

Then $Re_{kk} = (e_{kk}) \subseteq (A)$, and since $A \in Re_{kk}$ implies that $(A) \subseteq Re_{kk}$, we conclude that $(A) = Re_{kk}$ and so Re_{kk} is simple.

Problem 3. (After Chapter 14.1.) (Exercise 14.1.3 in the book.) Let M be an additive abelian group. Show that there is only one way of making it a $\mathbb{Z}$-module.

Solution. We want to equip M with the structure of a $\mathbb{Z}$-module. Clearly we have that $0m = 0$, that $1m = m$ and that $(-1)m = -m$ hold for all $m \in M$ (these hold for any module). Let $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. If $k > 0$, then since distributivity must hold in M we have that for any $m \in M$

$$km = (\underbrace{1 + \cdots + 1}_{k \text{ times}})m = \underbrace{1m + \cdots + 1m}_{k \text{ times}} = \underbrace{m + \cdots + m}_{k \text{ times}}.$$

Hence, for any $m \in M$, we have to define km to be equal to the element $\underbrace{m + \cdots + m}_{k \text{ times}}$ in M. Similarly, if $k < 0$, then we have that for any $m \in M$

$$km = (\underbrace{-1 - \cdots - 1}_{-k \text{ times}})m = \underbrace{(-1)m + \cdots + (-1)m}_{-k \text{ times}} = \underbrace{-m - \cdots - m}_{-k \text{ times}}.$$

Hence, for any $m \in M$, we have to define km to be equal to the element $\underbrace{-m - \cdots - m}_{-k \text{ times}}$ in M. We conclude that there is a unique way to give a $\mathbb{Z}$-module structure to M.

Problem 4. (After Chapter 11.3.) Show that $\mathbb{Z}[X]$ is not a PID but is a UFD.

Solution. By Example 8.8(1) we have that $\mathbb{Z}$ is a UFD and so by Example 8.8(3) we have that $\mathbb{Z}[X]$ is a UFD.

We now claim that $\mathbb{Z}[X]$ is not a PID. Consider the ideal $(2, X) \subseteq \mathbb{Z}[X]$. Assume to a contradiction that $(2, X) = (p(X))$ for some $p(X) \in \mathbb{Z}[X]$. Since $\mathbb{Z}[X]$ is commutative and unital, we have that

$$(p(X)) = \{p(X)q(X) \mid q(X) \in \mathbb{Z}[X]\}.$$

Then $2 \in (2, X) = (p(X))$ implies that $2 = p(X)q(X)$ for some $q(X) \in \mathbb{Z}[X]$. Hence $p(X) = \pm 1$ or $p(X) = \pm 2$. Then $p(X) \in (2, X)$ implies that $p(X) = 2f(X) + Xg(X)$ for some $f(X), g(X) \in \mathbb{Z}[X]$. If $p(X) = \pm 1$ this is impossible so we conclude that $p(X) = \pm 2$. Then $X \in (2, X) = (p(X))$ gives that $X = \pm 2q(X)$ for some $q(X) \in \mathbb{Z}[X]$, which is impossible. We conclude that no such $p(X)$ exists and so the ideal $(2, X)$ is not principal.

Problem 5. (After Chapter 14.5.) (Exercise 14.5.5 in the book.) Show that ${}_{\mathbb{Z}}\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Solution. Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, where $p, q, r, s \in \mathbb{Z}$. Then $qr, -ps \in \mathbb{Z}$ and so

$$qr\frac{p}{q} + (-ps)\frac{r}{s} = rp - pr = 0.$$

Hence the elements $\frac{p}{q}$ and $\frac{r}{s}$ are not linearly independent. Now assume to a contradiction that ${}_{\mathbb{Z}}\mathbb{Q}$ is free, that is, that ${}_{\mathbb{Z}}\mathbb{Q}$ has a $\mathbb{Z}$-basis. Then this basis is a linearly independent set, and hence it has to contain exactly one element, say $\frac{a}{b}$ with $a, b \in \mathbb{Z}$. Then $\{\frac{a}{b}\}$ generates ${}_{\mathbb{Z}}\mathbb{Q}$ and so there exists $k \in \mathbb{Z}$ such that

$$\frac{a}{2b} = k\frac{a}{b}.$$

This gives $1 = 2k$, contradicting that $k \in \mathbb{Z}$. Hence such a basis does not exist and so ${}_{\mathbb{Z}}\mathbb{Q}$ is free.

Problem 6. (After Chapter 14.5.)

(a) (Exercise 14.5.6 in the book.) Show that every ideal of $\mathbb{Z}$ is free as a $\mathbb{Z}$-module.

(b) (Exercise 14.5.7 in the book.) Show that every principal left ideal in a unital integral domain R is free as a left R-module.

Solution.

(a) Let I be an ideal in $\mathbb{Z}$. Since $\mathbb{Z}$ is a PID we have that $I = (n)$ for some $n \in \mathbb{Z}$. If $n = 0$, then $I = 0$ is free with basis given by the empty set $\varnothing$. If $n \neq 0$, then $I = (n)$ implies that $\{n\}$ generates I. Moreover, if $kn = 0$ for some $k \in \mathbb{Z}$, then $k = 0$. Therefore $\{n\}$ is a linearly independent set. Hence in this case $\{n\}$ is a basis of I.

(b) Let I be a left ideal in R. Since R is a PID we have that $I = (a)$ for some $a \in R$. If $a = 0$, then $I = 0$ is free with basis given by the empty set $\varnothing$. If $a \neq 0$, then $I = (a)$ implies that $\{a\}$ generates I. Moreover, if $ra = 0$ for some $r \in R$, then $r = 0$ since R is an integral domain and $a \neq 0$. Therefore $\{a\}$ is a linearly independent set. Hence in this case $\{a\}$ is a basis of I.

Problem 7. (After Chapter 14.2.) (Exercise 14.2.9 in the book.) Let R be the ring of all 2×2 upper triangular matrices over the field $\mathbb{Z}_2$.

(a) List all direct summands of R as a left R-module; that is, all left ideals A of the ring R such that $A \oplus B = R$ for some left ideal B of R. *(Hint: see Example 5.4 in the notes)*

(b) For each direct summand in (a) list all idempotents generating it as a left R-module.

Solution.

(a) Assume that $R = A \oplus B$ for some left ideals A and B. By Example 5.4(2) we have that there exists an orthogonal family of idempotents $\{e_1, e_2\}$ such that $A = Re_1$ and $B = Re_2$. Let us find the idempotents of R. The elements of R are

$$r_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad r_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$r_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad r_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_7 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad r_8 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The idempotents are $r_1, r_2, r_4, r_5, r_6, r_7$. We have

$$Rr_1 = 0, \quad Rr_2 = \{r_1, r_2, r_3, r_4\} = Rr_4, \quad Rr_5 = \{r_1, r_5\}, \quad Rr_6 = R, \quad Rr_7 = \{r_1, r_7\}.$$

Hence if A is a direct summand of R as a left R-module, we have that $A \in \{0, Rr_2, Rr_5, R, Rr_7\}$. It is clear that 0 and R are direct summands of R as $R = R \oplus 0$. Moreover we have that

$$1_R = r_6 = r_2 + r_5 \in Rr_2 + Rr_5,$$

and so $R = Rr_2 + Rr_5$ (because 1_R is in this ideal). By the above description we compute that $Rr_2 \cap Rr_5 = 0$ and so we have that $R = Rr_2 \oplus Rr_5$. Hence also Rr_2 and Rr_5 are direct summands of R. Similarly, we have that

$$1_R = r_6 = r_7 + r_4 \in Rr_7 + Rr_4$$

and $Rr_7 \cap Rr_4 = 0$, which shows that Rr_7 is also a direct summand of R (we have $Rr_4 = Rr_2$ so we do not obtain a new direct summand for Rr_4). We conclude that the left R-modules which are direct summands of R are $\{0, R, Rr_2, Rr_5, Rr_7\}$.

(b) We have done this already in part (a). We have that r_1 generates 0, that r_2 and r_4 generate the same ideal $\{r_1, r_2, r_3, r_4\}$, that r_5 generates $\{r_1, r_5\}$, that r_6 generates R and that r_7 generates $\{r_1, r_7\}$.

Problem 8. (After Chapter 14.3.) Let R be a unital ring and let M be a left R-module. Let $X \subseteq M$ be a subset.

(a) Show that
$$(X) = \{r_1x_1 + \cdots + r_mx_m \mid m > 0, r_i \in R, x_i \in X\}.$$

(b) Let $f : M \to M'$ be a left R-module homomorphism. Show that if $M = (X)$, then $\operatorname{Im} f = (f(X))$.

Solution.

(a) Let us set
$$N = \{r_1x_1 + \cdots + r_mx_m \mid m > 0, r_i \in R, x_i \in X\}.$$

We first claim that N is a left R-module. Clearly $0 \in N$ and so $N \neq \varnothing$. Let $x = r_1x_1 + \cdots + r_mx_m \in N$, $y = s_1y_1 + \cdots + s_ny_n \in N$ and $r \in R$. Then

$$x - y = (r_1x_1 + \cdots + r_mx_m) - (s_1y_1 + \cdots + s_ny_n) = r_1x_1 + \cdots + r_mx_m - s_1y_1 - \cdots - s_ny_n \in N,$$

and
$$rx = r(r_1x_1 + \cdots + r_mx_m) = (rr_1)x_1 + \cdots + (rr_m)x_m \in N,$$

which shows that N is a left R-module. Moreover, we have that if $x = r_1x_1 + \cdots + r_mx_m \in N$, then $x_1, \ldots, x_m \in X$ and $r_1, \ldots, r_m \in R$. Since (X) is a left R-module, it follows that

$$x = r_1x_1 + \cdots + r_mx_m \in (X),$$

and so $N \subseteq (X)$. Since $X \subseteq N$, we also have $(X) \subseteq N$ from which we obtain that $(X) = N$ as required.

(b) We have that $X \subseteq M$ and so $f(X) \subseteq f(M) = \operatorname{Im} f$ which shows that $(f(X)) \subseteq \operatorname{Im} f$. On the other hand, let $b \in \operatorname{Im} f$. Then $b = f(a)$ for some $a \in M$. Since $M = (X)$, we have by part (a) that

$$a = r_1 x_1 + \cdots + r_m x_m,$$

for some $x_1, \ldots, x_m \in X$ and $r_1, \ldots, r_m \in R$. Then

$$f(a) = f(r_1 x_1 + \cdots + r_m x_m) = r_1 f(x_1) + \cdots + r_m f(x_m)$$

is in $(f(B))$ since $f(x_1), \ldots, f(x_m) \in f(X)$ and $r_1, \ldots, r_m \in R$. Therefore $b = f(a) \in (f(B))$ and so we have shown that $\operatorname{Im} f \subseteq (f(B))$. This shows that $(f(B)) = \operatorname{Im} f$ as required.

Problem 9. (After Chapter 14.3.) Let R be a unital ring and let $f : M \to N$ be a homomorphism of left R-modules. Let $B \subseteq M$ and let $f(B) = \{f(b) \mid b \in B\}$. Show that the following hold.

(a) Assume that f is injective. If B is linearly independent, then $f(B)$ is also linearly independent.

(b) Assume that f is surjective. If $M = (B)$, then $N = (f(B))$.

(c) Assume that f is bijective. Then B is a basis of M if and only if $f(B)$ is a basis of N.

Solution.

(a) Let $\{f(b_1), \ldots, f(b_n)\} \subseteq f(B)$ be a finite subset for some $b_1, \ldots, b_n \in B$. Assume that

$$r_1 f(b_1) + \cdots + r_n f(b_n) = 0. \tag{1}$$

To show that $f(B)$ is linearly independent, we have to show that $r_1 = \cdots = r_n = 0$. Since f is a homomorphism of left R-modules, (1) becomes

$$f(r_1 b_1 + \cdots + r_n b_n) = 0 = f(0),$$

and so by injectivity of f we obtain that

$$r_1 b_1 + \cdots + r_n b_n = 0.$$

Since $b_1, \ldots, b_n \in B$ and B is linearly independent, we obtain that $r_1 = \cdots = r_n$, as required.

(b) By Problem 8 we have that $\operatorname{Im} f = (f(B))$. Since f is surjective, we have that $\operatorname{Im} f = N$. Hence we conclude that $N = f(B)$.

(c) Assume that B is a basis of M. Then B is linearly independent and so $f(B)$ is linearly independent by (a) since f is injective. Moreover $M = (B)$ and so $N = (f(B))$ by (b) since f is surjective. Hence $f(B)$ is linearly independent and generates N, which shows that $f(b)$ is a basis of N.

Assume now that $f(B)$ is a basis of N. Since f is bijective, there exists a bijective homomorphism of left R-modules $f^{-1} : N \to M$ such that $f^{-} f = \mathrm{id}_M$. Hence by applying the first paragraph of this part of the solution to f^{-1}, we obtain that $f^{-1} f(B)$ is a basis of M. Since $f^{-1} f(B) = B$, the claim is shown.

Problem 10. (After Chapter 14.3.) Based on the second and third isomorphism theorems for rings (problems 8 and 9 in problem set 2) state the second and third isomorphism theorems for left modules over a unital ring R.

Solution. Second isomorphism theorem for modules: Let M be a left R-module. Let N and L be submodules of M. Then the following hold.

(a) The sum $N + L := \{n + l \mid n \in N, l \in L\}$ is a submodule of M.

(b) The intersection $N \cap L$ is a submodule of M.

(c) The quotient modules $(N + L)/L$ and $N/(N \cap L)$ are isomorphic.

Third isomorphism theorem for modules: Let M be a left R-module. Let N and L be submodules of M such that $N \subseteq L$. Then the quotient module $(M/N)/(L/N)$ is isomorphic to the quotient module M/L.

Problem 11. (After Chapter 14.4.) Let R be a unital ring.

(a) Let M be a semisimple module. Show that every nonzero submodule of M is semisimple. *(Hint: use Theorem 11.7(3) in the notes.)*

(b) (Exercise 14.4.2 in the book.) Let M be a semisimple left R-module. Suppose that N is a submodule of M with $N \neq M$. Prove that M/N is semisimple. *(Hint: use Theorem 11.7(3) in the notes to obtain that $M = N \oplus L$ for some submodule $L \subseteq M$. Show that $M/N \cong L$ using the first isomorphism theorem for modules, and then use part (a).)*

(c) (Exercise 14.4.7 in the book.) Show that ${}_RR$ is semisimple as a left R-module if and only if every left R-module is semisimple. *(Hint: For the direction ${}_RR$ semisimple implies that every left R-module is semisimple, show that if M is a left R-module, then (m) is semisimple for any $m \in M$ by using part (b).)*

Solution.

(a) Recall that by Theorem 11.7(3) we know that a left R-module U is semisimple if and only if for every submodule $X \subseteq U$ there exists a submodule $Y \subseteq U$ such that $U = X \oplus Y$.

Let $N \subseteq M$ be a nonzero submodule of M and we show that N is semisimple. Since M is semisimple, by Theorem 11.7(3) there exists a submodule $L \subseteq M$ such that $M = N \oplus L$. To show that N is semisimple, let $K \subseteq N$ be a submodule and we need to find a submodule $O \subseteq N$ such that $N = K \oplus O$. Since $K \subseteq N \subseteq M$ and since M is semisimple, we have that there exists a submodule $P \subseteq M$ such that $M = K \oplus P$. Hence

$$N \oplus L = M = K \oplus P.$$

Now set $O = P \cap N$. Clearly $O \subseteq N$ is a submodule of N (as the intersection of submodules is a submodule). We claim that $K \oplus O = N$. Let us first show that $K + O = N$. If $n \in N$, then $n \in M = K \oplus P$ and so $n = k + p$ for some $k \in K$ and $p \in P$. Then $p = n - k$ with $n \in N$ and $k \in K \subseteq N$. Hence $p \in N$ as well and so $p \in P \cap N = O$. Therefore we have that $n = k + p$ with $k \in K$ and $p \in O$, and so $n \in K + O$. This shows that $N \subseteq K + O$. Since both $K \subseteq N$ and $O \subseteq N$ hold, we also have that $K + O \subseteq N$. We conclude that $K + O = N$. It remains to show that $K \cap O = \{0\}$. We have

$$K \cap O = K \cap (P \cap N) = (K \cap P) \cap N \subseteq K \cap P = \{0\},$$

where the last equality follows since the sum $K \oplus P$ is direct. Therefore $K \cap O = \{0\}$ and so $N = K \oplus O$, which shows that N is semisimple.

(b) Since M is semisimple, we have by Theorem 11.7(3) that there exists a submodule $L \subseteq M$ such that $M = N \oplus L$. We claim that $M/N \cong L$. Consider the R-module homomorphism $\pi_L : N \oplus L \to L$ given by $\pi_L(n + l) = l$ for any $n + l \in N \oplus L$. It is surjective since for every $l \in L$ we have that $\pi_L(0 + l) = l$. Hence $\operatorname{Im} \pi_L = L$. Moreover, we claim that $\operatorname{Ker} \pi_L = N$. To show this, let $x = n + l \in \operatorname{Ker} \pi_L$. Then $\pi_L(x) = 0$ gives $\pi_L(n + l) = 0$ or $l = 0$. Hence $x = n \in N$. Therefore $\operatorname{Ker} \pi_L \subseteq N$. Clearly if $n \in N$, then $n = n + 0$ and so $\pi_L(n) = \pi_L(n + 0) = 0$, which shows that $n \in \operatorname{Ker} \pi_L$. Therefore, $N \subseteq \operatorname{Ker} f$ as well which shows that $N = \operatorname{Ker} \pi_L$. By the first isomorphism theorem for modules we obtain that

$$M/N = (N \oplus L)/\operatorname{Ker} \pi_L \cong \operatorname{Im} \pi_L = L.$$

Hence indeed we have that $M/N \cong L$. But L is a submodule of M and so it is semisimple by part (a). It follows that M/N is semisimple as well.

(c) Assume first that every left R-module is semisimple. Then ${}_RR$ is a left R-module and so it is also semisimple.

Assume now that ${}_RR$ is semisimple as a left R-module. Let M be a left R-module and we show that M is semisimple as well. Let $m \in M$. We claim that (m) is a semisimple R-module. Indeed, consider the left multiplication map $f : R \to (m)$ given by $f(r) = rm$. For $r_1, r_2, r \in R$ we have that

$$f(r_1 + r_2) = (r_1 + r_2)m = r_1m + r_2m = f(r_1) + f(r_2),$$

and

$$f(rr_1) = (rr_1)m = r(r_1m) = rf(r_1m),$$

and so f is a homomorphism of left R-modules. Since R is unital, we have that $(m) = Rm = \{rm \mid r \in R\}$ by Proposition 9.11. For any $rm \in (m)$ we have that $f(r) = rm$ and so f is surjective or $\operatorname{Im} f = (m)$. Let $N = \operatorname{Ker} f$. Then N is a submodule of ${}_RR$ and so by the first isomorphism theorem for modules we obtain that

$$R/\operatorname{Ker} f \cong \operatorname{Im} f = (m).$$

If $\operatorname{Ker} f = R$ then $0 = R/\operatorname{Ker} f = (m)$ and so (m) is semisimple. If $\operatorname{Ker} f \neq R$, then by part (b) we know that $R/\operatorname{Ker} f$ is semisimple. Hence in any case (m) is semisimple and so it is the sum of a family of simple modules by Theorem 11.7(1). Then

$$M = \sum_{m \in M} (m)$$

where each (m) is the sum of a family of simple modules, and so M itself is the sum of a family of simple modules. It follows that M is semisimple by Theorem 11.7(1).

Problem 12. (After Chapter 14.3.) Let R be a ring and let M be a left R-module. Let N, L, L' be submodules of M such that $N \oplus L = N \oplus L'$.

(a) (Exercise 14.2.8 in the book.) Show that it is not necessarily true that $L = L'$.

(b) Show that $(N \oplus L)/N \cong L$.

(c) (Exercise 14.3.6 in the book.) Show that $L \cong L'$.

Solution.

(a) This is not true even for vector spaces. Indeed, let $M = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Then M is an $\mathbb{R}$-module (this is the usual 2-dimensional vector space $\mathbb{R}^2$). Let $N = \{(x, 0) \mid x \in \mathbb{R}\}$, $L = \{(0, y) \mid y \in \mathbb{R}\}$ and $L' = \{(x, x) \mid x \in \mathbb{R}\}$. Clearly, N, L and L' are all submodules of $\mathbb{R}^2$ (these are the lines $y = 0$, $x = 0$ and $x = y$ in $\mathbb{R}^2$). Moreover, it is also clear that $N \cap L = \{(0, 0)\}$ and $N \cap L' = \{(0, 0)\}$. Hence the sums of N with L and of N with L' are direct. Now let $(x, y) \in M = \mathbb{R}^2$. We have

$$(x, y) = \underbrace{(x, 0)}_{\in N} + \underbrace{(0, y)}_{\in L},$$

and so $M = N \oplus L$. We also have

$$(x, y) = \underbrace{(x - y, 0)}_{\in N} + \underbrace{(y, y)}_{\in L'},$$

and so $M = N \oplus L'$ as well. Hence we have $N \oplus L = N \oplus L'$, but $L \neq L'$ since $(1, 1) \in L'$ but $(1, 1) \notin L$.

(b) Define a map $\phi : N \oplus L \to L$ by $n + l \mapsto l$. We claim that ϕ is well-defined. Indeed assume that $n + l = n' + l' \in N \oplus L$. Then $\phi(n + l) = l$ and $\phi(n' + l') = l'$ and we need to show that $l = l'$. Since $n + l = n' + l'$, we have that $n - n' = l - l'$. Then $l - l' = n - n' \in N$ and $l - l' \in L$ and so $l - l' \in N \cap L = \{0\}$. Hence $l = l'$, as required.

Now we claim that ϕ is a homomorphism of left R-modules. Let $n_1 + l_1, n_2 + l_2 \in N \oplus L$ and $r \in R$. We have

$$\phi((n_1 + l_1) + (n_2 + l_2)) = \phi((n_1 + n_2) + (l_1 + l_2)) = l_1 + l_2 = \phi(n_1 + l_1) + \phi(n_2 + l_2),$$

and

$$\phi(r(n_1 + l_1)) = \phi(rn_1 + rl_1) = rl_1 = r\phi(n_1 + l_1),$$

which shows that ϕ is a homomorphism of left R-modules. We now claim that $\operatorname{Im}\phi = L$. Let $l \in L$. Then $\phi(0 + l) = l$ and so ϕ is surjective, which shows that $\operatorname{Im}\phi = L$. Next, we claim that $\operatorname{Ker}\phi = N$. If $n \in N$, then $n = n + 0 \in N \oplus L$ and $\phi(n + 0) = 0$ so $n \in N$. This shows that $N \subseteq \operatorname{Ker}\phi$. On the other hand, assume that $n + l \in \operatorname{Ker}\phi$, that is $\phi(n + l) = 0$. Since $\phi(n + l) = l$, we obtain that $l = 0$ and so $n + l = n \in N$. Hence $\operatorname{Ker}\phi \subseteq N$ and so $\operatorname{Ker}\phi = N$.

We have shown that $\phi : N \oplus L \to L$ is a homomorphism of left R-modules with $\operatorname{Im}\phi = L$ and $\operatorname{Ker}\phi = N$. By the first isomorphism theorem for modules we obtain

$$(N \oplus L)/N = (N \oplus L)/\operatorname{Ker}\phi \cong \operatorname{Im}\phi = L,$$

as required.

(c) By part (b) and using the assumption $N \oplus L = N \oplus L'$ we have that

$$L \cong (N \oplus L)/N = (N \oplus L')/N \cong L',$$

as required.

Problem 13. (After Chapter 14.3.) (Exam August 2013, problem 5.) Let R, S be unital rings and $\phi : S \to R$ be a ring homomorphism such that $\phi(1_S) = 1_R$.

(a) What does it mean for a left R-module to be finitely generated?

(b) (i) Explain how we can view every left R-module as a left S-module. *(Hint: in other words, if ${}_RM$ is a left R-module, give to M the structure of a left S-module.)*

(ii) Assume that ϕ is surjective. Show that a finitely generated left R-module is also a finitely generated left S-module.

(c) (i) Write down Zorn's lemma.

(ii) Show that every nonzero finitely generated left R-module has a maximal submodule. *(If M is a left R-module, then a* maximal *submodule of M is a submodule $N \subseteq M$ with $N \neq M$ and such that if L is a submodule of M with $N \subseteq L$, then $N = L$ or $L = M$.)*

Solution.

(a) A left R-module M is finitely generated if there exist $m_1, \dots, m_n \in M$ such that $M = (m_1, \dots, m_n)$, that is, M is the smallest left R-module containing $\{m_1, \dots, m_n\}$.

(b) (i) Let M be a left R-module. We write ${}_RM$ when we want to emphasize the left R-module structure of M, we write M when we want to emphasize only the abelian group structure of M and we write ${}_SM$ when we want to emphasize the left S-module structure of M which we will soon define.

For every $m \in M$ and every $s \in S$ we define

$$sm := \phi(s)m.$$

We claim that this makes M into a left S-module ${}_SM$. By assumption, M is an abelian group. Using the fact that ϕ is a ring homomorphism, for every $m_1, m_2, m \in M$ and $s_1, s_2, s \in S$ we have

$$s(m_1 + m_2) = \phi(s)(m_1 + m_2) = \phi(s)m_1 + \phi(s)m_2 = sm_1 + sm_2,$$

$$(s_1 + s_2)m = \phi(s_1 + s_2)m = (\phi(s_1) + \phi(s_2))m = \phi(s_1)m + \phi(s_2)m = s_1m + s_2m,$$

$$(s_1s_2)m = \phi(s_1s_2)m = (\phi(s_1)\phi(s_2))m = \phi(s_1)(\phi(s_2)m) = \phi(s_1)(s_2m) = s_1s_2m,$$

$$1_Ss = \phi(1_S)s = 1_Rs = s,$$

which show that indeed ${}_SM$ is a left S-module.

(ii) Assume that ${}_RM = {}_R(m_1, \dots, m_n)$ for some $m_1, \dots, m_n \in {}_RM$ as a left R-module. We claim that ${}_SM = {}_S(m_1, \dots, m_n)$ as a left S-module. By Problem 8(a) we have that

$$ {}_R(m_1, \dots, m_n) = \{r_1m_1 + \cdots + r_nm_n \mid r_i \in R\}, $$

and

$$ {}_S(m_1, \dots, m_n) = \{s_1m_1 + \cdots + s_nm_n \mid s_i \in S\}. $$

Clearly we have ${}_S(m_1, \dots, m_n) \subseteq M$. For the other inclusion, let $m \in {}_SM$. Then $m \in M = M_R = {}_R(m_1, \dots, m_n)$ and so there exist $r_1, \dots, r_n \in R$ such that

$$ m = r_1m_1 + \cdots + r_nm_n. $$

Since ϕ is surjective, there exist $s_1, \dots, s_n \in S$ such that $\phi(s_i) = r_i$ for $1 \leq i \leq n$. Then

$$ m = \phi(s_1)m_1 + \cdots + \phi(s_n)m_n = s_1m_1 + \cdots + s_nm_n, $$

where the last equality follows by the way that multiplication is defined in ${}_SM$. Since $m = s_1m_1 + \cdots + s_nm_n \in {}_S(m_1, \dots, m_n)$, we conclude that $M \subseteq {}_S(m_1, \dots, m_n)$ and so $M = {}_S(m_1, \dots, m_n)$. Hence M is finitely generated.

(c) (i) Zorn's lemma: Let $(S, \leq)$ be a nonempty partially ordered set (poset). If every nonempty chain in S has an upper bound in S, then there exists a maximal element in S.

(ii) Let M be a finitely generated left R-module. Set

$$ S = \{L \text{ is a submodule of } M \mid L \neq M\}. $$

Since $0 \in S$, we have that $S \neq \varnothing$. We claim that $(S, \subseteq)$ is a poset. Clearly for any $L \in S$ we have $L \subseteq L$ and so $\subseteq$ is reflexive. For $L, L' \in S$ we have that $L \subseteq L'$ and $L' \subseteq L$ imply that $L = L'$ and so $\subseteq$ is antisymmetric. Finally for $L, L', L'' \in S$ we have that $L \subseteq L'$ and $L' \subseteq L''$ and so $L \subseteq L''$. Now let C be a nonempty chain in S, that is a nonempty subset $C \subseteq S$ such that for all $L, L' \in C$ one of $L \subseteq L'$ and $L' \subseteq L$ holds. We claim that

$$ U = \bigcup_{L \in C} L $$

is a submodule of M. Since $C \neq \varnothing$, we have that $U \neq \varnothing$. Let $x, y \in U$ and $r \in R$. Then there exist $L_1, L_2 \in C$ such that $x \in L_1$ and $y \in L_2$. Since C is a chain, one of $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$ holds. Without loss of generality we assume that $L_1 \subseteq L_2$. Then $x, y \in L_2$ and so

- $x - y \in L_2 \subseteq U$ implies $x - y \in U$, and
- $rx \in L_2 \subseteq U$ implies $rx \in U$.

Hence U is a submodule of M. Next we claim that $U \neq M$. Indeed, assume to a contradiction that $U = M$. Since M is finitely generated, there exist $m_1, \dots, m_n \in M$ such that $U = M = (m_1, \dots, m_n)$. Hence $m_1, \dots, m_n \in U$. Then $m_i \in L_i$ for some $L_i \in C$. Since C is a chain, there exists $j \in \{1, \dots, n\}$ such that $L_i \subseteq L_j$ for all $i \in \{1, \dots, n\}$. Then $m_1, \dots, m_n \in L_j$ and so

$$ M = (m_1, \dots, m_n) \subseteq L_j \subseteq M, $$

which implies that $L_j = M$. But this contradicts $L_j \in S$. Hence we indeed have that $U \neq M$. Therefore $U \in S$. and U is an upper bound of C by construction. By Zorn's lemma there exists a maximal element $N \in S$. Since $N \subseteq M$, it is enough to show that N is a maximal submodule of M. Assume $N \subseteq L$ for some submodule L of M with $L \neq M$. Then $L \in S$ and so $N = L$ since N is a maximal element. Hence N is indeed a maximal submodule of M.

Problem 14. (After Chapter 14.4.) (Exam December 2016, problem 4.) Let R be a unital ring. Let M be a left R-module and let S be a simple submodule of M such that M/S is also simple. Let N be a submodule of M.

(a) Show that one of $N = 0$, $N = S$, $N = M$ and $N \cong M/S$ holds. *(Hint: use the first isomorphism theorem for modules)*

(b) Show that if $N \neq S$ and N is simple, then $N \cong M/S$ and $M \cong N \oplus S$.

(c) Give an explicit example of a unital ring R and left R-modules M, S and N as above, such that $N = S$ and $N \cong M/S$, but $M \ncong N \times S$. *(Hint: consider $R = F[X]/(X^2)$ for some field F, $M = {}_RR$ and $N = (\overline{X})$.)*

Solution.

(a) Assume first that $0 \subseteq N \subseteq S$. Since S is simple, we conclude that $N = 0$ or $N = S$. From now on we may assume that $N \not\subseteq S$. In particular, we may also assume that $M \neq S$ (since $N \subseteq M$ holds).

Let $\iota_N : N \to M$ be the inclusion of N, and let $\pi : M \to M/S$ be the projection map, that is $\pi(m) = m+S$ for any $m \in M$. Then we obtain a map $f := \pi \circ \iota_N : N \to M/S$. This is a composition of R-module homomorphisms and so it is an R-module homomorphism. In particular $\operatorname{Im} f$ is a submodule of M/S. Since M/S is simple, we have that $\operatorname{Im} f = 0$ or $\operatorname{Im} f = M/S$. Since $N \not\subseteq S$, there exists $n \in N \setminus S$. Then $f(n) = \pi \circ \iota_N(n) = n + S \in \operatorname{Im} f$ is nonzero because $n \notin S$. Then $\operatorname{Im} f \neq 0$ and so we conclude that $\operatorname{Im} f = M/S$. Next we claim that $\operatorname{Ker} f = S \cap N$. Indeed if $f(n) = 0 + S$ for some $n \in N$, then $n + S = 0 + S$ and so $n \in S$. Hence $n \in S \cap N$ and so $\operatorname{Ker} f \subseteq S \cap N$. On the other hand, if $n \in S \cap N$, then $f(n) = n + S = 0 + S$ since $n \in S$ and so $S \cap N \subseteq \operatorname{Ker} f$. Therefore $\operatorname{Ker} f = S \cap N$. By the first isomorphism theorem for modules we obtain that

$$N/(N \cap S) = N/\operatorname{Ker} f \cong \operatorname{Im} f = M/S$$

Since $0 \subseteq N \cap S \subseteq S$ and S is simple, we have that $N \cap S = 0$ or $N \cap S = S$. If $N \cap S = 0$, then the above gives

$$N \cong M/S,$$

which is one of the four possibilities. Next assume that $N \cap S = S$. Then we have that $S \subseteq N$. By the correspondence theorem (theorem 10.6 in the notes) applied to the homomorphism of left R-modules $\pi : M \to M/S$, there is a one-to-one correspondence between submodules of M containing $\operatorname{Ker} \pi = S$ and submodules of $\operatorname{Im} f = M/S$. Since M/S is simple, it follows that the only submodules of M/S are 0 and M/S (and these are different since $M \neq S$). Therefore, there are exactly two submodules of M containing S, and these have to be M and S. Hence if $S \subseteq N$, we obtain that $N = S$ or $N = M$, which completes the proof.

(b) Since $N \neq S$, following part (a) we obtain an isomorphism $N/(N \cap S) \cong M/S$ of left R-modules. We claim that $N \cap S = 0$. Indeed, assume to a contradiction that $N \cap S \neq 0$. Then $N \cap S \subseteq S$ implies that $N \cap S = S$ since S is simple. Similarly, $N \cap S \subseteq N$ implies that $N \cap S = N$ since N is simple. Hence $N = S$, contradicting the assumption $N \neq S$. Therefore $N \cap S = 0$ and hence

$$N \cong N/0 = N/(N \cap S) \cong M/S,$$

as required.

Next we show that $M \cong N \oplus S$. We have already seen that $N \cap S = 0$ and so it remains to show that $M = N + S$. Clearly $N + S \subseteq M$ and so we only need to show that $M \subseteq N + S$. Consider the map $f = \pi \circ \iota_N : N \to M/S$ from part (a), where $\pi : M \to M/S$ is the projection map and $\iota_N : N \to M$ is the inclusion of N into M. We have already seen that this map is an isomorphism. Let $f^{-1} : M/S \to N$ be its inverse. Then for every $m \in M$ we have that $f^{-1} \circ \pi(m) \in N$. We claim that $m - f^{-1} \circ \pi(m) \in S$. First notice that since $f^{-1} \circ \pi(m) \in N$, we have that $f^{-1} \circ \pi(m) = \iota_N \circ f^{-1} \circ \pi(m)$ since ι_N is just the inclusion map from N to M. Then we compute

$$\begin{aligned}
\pi(m - f^{-1} \circ \pi(m)) &= \pi(m - \iota_N \circ f^{-1} \circ \pi(m)) \\
&= \pi(m) - \pi \circ \left(\iota_N \circ f^{-1} \circ \pi\right)(m) \\
&= \pi(m) - (\pi \circ \iota_N) \circ f^{-1} \circ \pi(m) \\
&= \pi(m) - f \circ f^{-1} \circ \pi(m) \\
&= \pi(m) - \pi(m) = 0.
\end{aligned}$$

Therefore

$$0 = \pi(m - f^{-1} \circ \pi(m)) = \big(m - f^{-1} \circ \pi(m)\big) + S \in M/S,$$

and so $m - f^{-1} \circ \pi(m) \in S$. Hence we can write

$$m = \underbrace{f^{-1} \circ \pi(m)}_{\in N} + \underbrace{m - f^{-1} \circ \pi(m)}_{\in S},$$

and since m was arbitrary, this shows that $M \subseteq N + S$. We conclude that $M \cong N \oplus S$ as required.

(c) Let $R = F[X]/(X^2)$ where F is a field. That is, elements of R are of the form $\overline{p(X)} = \overline{a + bX}$ with $a, b \in F$ (since if $n \geq 2$, then $X^n \in (X^2)$ and so X^n is equal to 0 in R). Moreover, R is commutative and so submodules of R are the same as ideals of R. Let $M = R$ as an R-module. We have that $\overline{X} \in R = M$ and so

$$\begin{aligned}(\overline{X}) &= \{(\overline{a + bX})\overline{X} \mid a, b \in F\} \\ &= \{\overline{aX + bX^2} \mid a, b \in F\} \\ &= \{\overline{aX} \mid a \in F\}\end{aligned}$$

is a submodule of M. We set $N = S = (\overline{X})$. To show that N is simple, we use Theorem 11.5(2). Hence we need to show that $N \neq 0$ and that for every $n \in N \setminus \{0\}$ we have that $N = (n)$. Clearly $N \neq 0$. Let $n \in N \setminus \{0\}$. Then $n = (\overline{f + eX})\overline{X} = \overline{fX}$ for some $f \in F$. Then $(n) \subseteq \overline{X}$. Moreover, we have that

$$\overline{X} = \overline{f^{-1}}\,\overline{fX} = \overline{f^{-1}}n \in (n),$$

and so $\overline{X} \in (n)$. Thus $(\overline{X}) \subseteq (n)$ as well which shows that $(n) = (\overline{X})$. Hence $N = (\overline{X})$ is a simple module.

Next we show that $N \cong M/S$, or $(\overline{X}) \cong R/(\overline{X})$ as R-modules. Elements of $R/(\overline{X})$ are now of the form $\overline{a}$ for $a \in F$, and the R-module structure in $R/\overline{X}$ is then given by

$$(\overline{c + dX})\overline{a} = \overline{ca} + \overline{daX} = \overline{ca},$$

for $\overline{a} \in R/\overline{X}$ and $\overline{c + dX} \in R$. Consider the map $g : (\overline{X}) \to R/(\overline{X})$ given by $\overline{aX} \mapsto \overline{a}$. We claim that g is a homomorphism of left R-modules. Indeed for any $\overline{aX}, \overline{bX} \in (\overline{X})$ and any $\overline{c + dX} \in R$ we have

$$g(\overline{aX} + \overline{bX}) = g(\overline{(a + b)X}) = \overline{a + b} = \overline{a} + \overline{b} = g(\overline{aX}) + g(\overline{bX}),$$

and

$$g((\overline{c + dX})\overline{aX}) = g(\overline{caX}) = \overline{ca} = \overline{ca} = (\overline{c + dX})\overline{a}.$$

Moreover, the map g is clearly bijective and so an isomorphism of R-modules. Hence we have shown that $N \cong M/S$.

Finally, it remains to show that $M \ncong N \oplus S$. We have that $M = R_R$ and so by Proposition 10.9 we have

$$R \cong \mathrm{End}_R(R_R) = \mathrm{End}_R(M).$$

A direct computation shows that the only idempotents in R are 0 and 1. But consider the morphism of R-modules $h : N \oplus S \to N \oplus S$ given by $(n, s) \mapsto (n, 0)$. Then clearly $h \neq 0$ and $h \neq \mathrm{id}_{N \oplus S}$, but

$$h^2(n, s) = h(h(n, s)) = h(n, 0) = (n, 0),$$

and so $h^2 = h$. Therefore $\mathrm{End}_R(N \oplus S)$ has more idempotents than $\mathrm{End}_R(M)$, and so these two rings cannot be isomorphic. It follows that M cannot be isomorphic to $N \oplus S$.

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 15. (After Chapter 14.3.) Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A $\mathbb{C}$-*representation* of Q is a collection $M_Q = (M_v, f_\alpha)_{v\in Q_0, \alpha\in Q_1}$ such that M_v is a vector space over $\mathbb{C}$ and $f_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $\mathbb{C}$-linear map.

Now set A_2 to be the quiver $A_2 = 1 \overset{\alpha}{\to} 2$. Recall that the path algebra $\mathbb{C}A_2$ is a $\mathbb{C}$-algebra with $\mathbb{C}$-basis given by $\{e_1, e_2, \alpha\}$.

(a) Let M be a left $\mathbb{C}A_2$-module. Set $M_1 = e_1M$, $M_2 = e_2M$ and $f_\alpha : e_1M \to e_2M$ given by $f_\alpha(x) = \alpha x$. Show that $F(M) := (M_i, f_\alpha)_{i=1,2}$ is a representation of A_2.

(b) Let $M_{A_2} = (M_i, f_\alpha)_{i=1,2}$ be a representation of A_2. Define a left $\mathbb{C}A_2$-module $G(M_{A_2})$ such that $G \circ F(M) \cong M$ for any left $\mathbb{C}A_2$-module M (it also satisfies $F \circ G(M_{A_2}) \cong M_{A_2}$, if one defines a suitable notion of morphisms of representations).

Solution.

(a) The only thing we need to show is that $f_\alpha : M_1 \to M_2$ is a $\mathbb{C}$-linear map. We have that $M_1 = e_1M$ and so its elements are of the form e_1m for $m \in M$. Let $e_1m_1, e_1m_2 \in e_1M$ and $k \in \mathbb{C}$. Then

$$\phi_\alpha(e_1m_1 + e_1m_2) = \alpha(e_1m_1 + e_1m_2) = \alpha e_1m_1 + \alpha e_1m_2 = \phi_\alpha(m_1) + \phi_\alpha(m_2),$$

and

$$\phi_\alpha(km_1) = \alpha(km_1) = k(\alpha m_1) = k\phi_\alpha(m_1),$$

which shows that ϕ_α is a linear map and so $F(M)$ is a representation of A_2.

(b) Since M_1 and M_2 are vector spaces, they are additive abelian groups and so the set $M = M_1 \times M_2$ is an additive abelian group (that is Definition 9.1(i) holds for M). For $(m_1, m_2) \in M$ we define

$$e_1(m_1, m_2) = (m_1, 0), \quad e_2(m_1, m_2) = (0, m_2), \quad \alpha(m_1, m_2) = (0, f_\alpha(m_1)).$$

We extend this to elements of $\mathbb{C}A_2$ using bilinearly. That is, if $x_1e_1 + x_2e_2 + x_\alpha\alpha \in \mathbb{C}A_2$, we set

$$(x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1, m_2) = (x_1m_1, x_2m_2 + x_\alpha f_\alpha(m_1)).$$

We claim that this gives to M the structure of a left $\mathbb{C}A_2$-module. For any $(m_1, m_2), (n_1, n_2) \in M$ and any $(x_1e_1 + x_2e_2 + x_\alpha\alpha), (y_1e_1 + y_2e_2 + y_\alpha\alpha), \in \mathbb{C}A_2$ we have

$$\begin{aligned}(x_1e_1 + x_2e_2 + x_\alpha\alpha)((m_1, m_2) + (n_1 + n_2)) &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1 + n_1, m_2 + n_2)\\ &= (x_1(m_1 + n_1), x_2(m_2 + n_2) + x_\alpha f_\alpha(m_1 + n_1))\\ &= (x_1m_1, x_2m_2 + x_\alpha f_\alpha(m_1)) + (x_1n_1, x_2n_2 + f_\alpha(n_1))\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1, m_2) + (x_1e_1 + x_2e_2 + x_\alpha\alpha)(n_1, n_2),\end{aligned}$$

which shows Definition 9.1(ii), we have

$$\begin{aligned}((x_1e_1 + x_2e_2 + x_\alpha\alpha) + (y_1e_1 + y_2e_2 + y_\alpha\alpha))(m_1, m_2) &= ((x_1 + y_1)e_1 + (x_2 + y_2)e_2 + (x_\alpha + y_\alpha)\alpha)(m_1, m_2)\\ &= ((x_1 + y_1)m_1, (x_2 + y_2)m_2 + (x_\alpha + y_\alpha(f_\alpha(m_1))\\ &= (x_1m_1, x_2m_2 + x_\alpha f_\alpha(m_1)) + (y_1m_1, y_2m_2 + y_\alpha f_\alpha(m_1))\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1, m_2) + (y_1e_1 + y_2e_2 + y_\alpha\alpha)(m_1, m_2),\end{aligned}$$

which shows Definition 9.1(iii), we have

$$\begin{aligned}((x_1e_1 + x_2e_2 + x_\alpha\alpha)(y_1e_1 + y_2e_2 + y_\alpha\alpha))(m_1, m_2) &= (x_1y_1e_1 + x_2y_2e_2 + (x_\alpha y_1 + x_2y_\alpha)\alpha)(m_1, m_2)\\ &= (x_1y_1m_1, x_2y_2m_2 + (x_\alpha y_1 + x_2y_\alpha)f_\alpha(m_1))\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)(y_1m_1, y_2m_2 + y_\alpha f_\alpha(m_1))\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)((y_1e_1 + y_2e_2 + y_\alpha\alpha)(m_1, m_2)),\end{aligned}$$

which shows Definition 9.1(iv), and, since $1_{\mathbb{C}A_2} = e_1 + e_2$, we also have

$$1_{\mathbb{C}A_2}(m_1, m_2) = (e_1 + e_2)(m_1, m_2) = (m_1, m_2),$$

which shows Definition 9.1(v). Therefore, this defines a left $\mathbb{C}A_2$-module $G(M_{A_2})$.

Now let M be a left $\mathbb{C}A_2$-module and we compute $G \circ F(M)$. By definition of F we have that $F(M) = (M_i, f_\alpha)_{i=1,2}$ where $M_i = e_iM$ and $f_\alpha : e_1M \to e_2M$ is given by $f_\alpha(x) = \alpha x$. Now by definition of G we have that $G(F(M)) = e_1M \times e_2M$ and the action of e_1, e_2 and α is given as above. Define a map $\phi : e_1M \times e_2M \to M$ via $\phi(m, n) \mapsto m + n$. We claim that this map is a bijective homomorphism of left $\mathbb{C}A_2$-modules. To see that this is a homomorphism of left $\mathbb{C}A_2$-modules, let $(m_1, m_2), (n_1, n_2) \in e_1M \times e_2M$ and $x_1e_1 + x_2e_2 + x_\alpha\alpha \in \mathbb{C}A_2$. Then

$$\begin{aligned}\phi((m_1, m_2) + (n_1, n_2)) &= \phi(m_1 + n_1, m_2 + n_2)\\ &= m_1 + n_1 + m_2 + n_2\\ &= m_1 + m_2 + n_1 + n_2\\ &= \phi(m_1, m_2) + \phi(n_1, n_2).\end{aligned}$$

Moreover, since $m_1 \in e_1M$ we have that $m_1 = e_1m_1$ and so $e_2e_1m_1 = 0$, and since $m_2 \in e_2M$ we have that $m_2 = e_2m_2$ and so $e_1e_2m_2 = 0$. Using these we obtain

$$\begin{aligned}\phi((x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1, m_2)) &= \phi(x_1m_1, x_2m_2 + x_\alpha f_\alpha(m_1))\\ &= x_1m_1 + x_2m_2 + x_\alpha f_\alpha(m_1)\\ &= x_1m_1 + x_2m_2 + x_\alpha\alpha m_1\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)(m_1 + m_2)\\ &= (x_1e_1 + x_2e_2 + x_\alpha\alpha)\phi(m_1, m_2).\end{aligned}$$

This shows that ϕ is a homomorphism of left $\mathbb{C}A_2$-modules. Next we claim that it is injective. Assume that $\phi(m_1, m_2) = 0$. Then $m_1 + m_2 = 0$ with $m_1 = e_1m_1$ and $m_2 = e_2m_2$. Therefore

$$0 = e_10 = e_1(m_1 + m_2) = e_1m_1 + e_1e_2m_2 = m_1 + 0 = m_1,$$

and so $m_1 = 0$ and similarly $m_2 = 0$. We conclude that $(m_1, m_2) = (0, 0)$ and so $\operatorname{Ker}\phi = \{0\}$. This shows that ϕ is injective. Finally, to see that ϕ is surjective, let $m \in M$. We have

$$\phi(e_1m, e_2m) = e_1m + e_2m = (e_1 + e_2)m = 1_{\mathbb{C}A_2}m = m,$$

and so ϕ is indeed surjective. Hence ϕ is an isomorphism of left $\mathbb{C}A_2$-modules, as required.

Problem 16. (After Chapter 14.3.) Let R, S be unital rings and $\phi : R \to S$ be a ring homomorphism such that $\phi(1_R) = 1_S$. Let M be a right R-module and N be a right S-module.

(a) Show that N can be made into a right R-module N_R by setting $nr := n\phi(r)$ for every $n \in N$ and every $r \in R$.

(b) By (a) we may view S as a right R-module. Show that the set $\operatorname{Hom}_R(S_R, M)$ has the structure of a right S module.

(c) Provide a bijection between the sets $\operatorname{Hom}_R(N_R, M)$ and $\operatorname{Hom}_S(N, \operatorname{Hom}_R(S_R, M))$.

Solution.

(a) Since N is a right S-module, it is an additive abelian group. Now let $r, r_1, r_2 \in R$ so that $\phi(r), \phi(r_1), \phi(r_2) \in S$, and let $n, n_1, n_2 \in N$. Using the fact that N is a right S-module, we check the axioms for N to be a right R-module:

- $(n_1 + n_2)r = (n_1 + n_2)\phi(r) = n_1\phi(r) + n_2\phi(r) = n_1r + n_2r$,
- $n(r_1 + r_2) = n\phi(r_1 + r_2) = n(\phi(r_1) + \phi(r_2)) = n\phi(r_1) + n\phi(r_2) = nr_1 + nr_2$,
- $n(r_1r_2) = n\phi(r_1r_2) = n(\phi(r_1)\phi(r_2)) = (n\phi(r_1))\phi(r_2) = (nr_1)r_2$,
- $n1_R = n\phi(1_R) = n1_S = n$.

We conclude that N is indeed a right R-module.

(b) Let $f \in \mathrm{Hom}_R(S_R, M)$, that is f is a homomorphism of right R-modules. Let $s \in S$. We define $f \cdot s : S_R \to M$ to be the map

$$(f \cdot s)(x) = f(sx).$$

First we have that $sx \in S$ and so indeed $f(sx) \in M$. Hence f is well-defined. Now we claim that $f \cdot s \in \mathrm{Hom}_R(S_R, M)$, that is that it is a homomorphism of right R-modules. Let $x, x_1, x_2 \in S_R$ and $r \in R$. Using the fact that f is a homomorphism of right R-modules and the right R-module structure of S, we check the axioms for $f \cdot s$ to be a homomorphism of right R-modules:

- $(f \cdot s)(x_1 + x_2) = f(s(x_1 + x_2)) = f(sx_1 + sx_2) = f(sx_1) + f(sx_2) = (f \cdot s)(x_1) + (f \cdot s)(x_2)$,
- $(f \cdot s)(xr) = f(s(xr)) = f(s(x\phi(r))) = f((sx)\phi(r)) = f((sx)r) = f(sx)r = (f \cdot s)(x)r$.

Now, since S_R and M are additive abelian groups, the set $\mathrm{Hom}_R(S_R, M)$ is itself an additive abelian group with group addition given by

$$(f_1 + f_2)(s) = f_1(s) + f_2(s)$$

for any $f_1, f_2 \in \mathrm{Hom}_R(S_R, M)$, $s \in S_R$. Now let $s, s_1, s_2 \in S$, let $f, f_1, f_2 \in \mathrm{Hom}_R(S_R, M)$ and let $x \in S_R$. Using the fact that f, f_1, f_2 are homomorphisms of right R-modules and the right R-module structure of S_R, we check the axioms for $\mathrm{Hom}_R(S_R, M)$ to be a right S-module:

- $((f_1 + f_2) \cdot s)(x) = (f_1 + f_2)(sx) = f_1(sx) + f_2(sx) = (f_1 \cdot s)(x) + (f_2 \cdot s)(x) = (f_1 \cdot s + f_2 \cdot s)(x)$,
- $(f \cdot (s_1 + s_2))(x) = f((s_1 + s_2)x) = f(s_1x + s_2x) = f(s_1x) + f(s_2x) = (f \cdot s_1)(x) + (f \cdot s_2)(x)$,
- $(f \cdot (s_1s_2))(x) = f((s_1s_2)x) = f(s_1(s_2x)) = (f \cdot s_1)(s_2x) = ((f \cdot s_1) \cdot s_2)(x)$,
- $(f \cdot 1_S)(x) = f(1_Sx) = f(x)$.

Since the above hold for arbitrary $x \in S_R$, we obtain that

- $(f_1 + f_2) \cdot s = f_1 \cdot s + f_2 \cdot s$,
- $f \cdot (s_1 + s_2) = f \cdot s_1 + f \cdot s_2$,
- $f \cdot (s_1s_2) = (f \cdot s_1) \cdot s_2$,
- $f \cdot 1_S = f$,

which shows that $\mathrm{Hom}_R(S_R, M)$ is indeed a right S-module.

(c) We start by defining a map

$$\Psi : \mathrm{Hom}_R(N_R, M) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(S_R, M)).$$

Let $f \in \mathrm{Hom}_R(N_R, M)$. Then $\Psi(f) \in \mathrm{Hom}_S(N, \mathrm{Hom}_R(S_R, M))$. Hence $\Psi(f)$ is a morphism of right S-modules

$$\Psi(f) : N \to \mathrm{Hom}_R(S_R, M),$$

which we define for $n \in N$ by

$$(\Psi(f))(n) = f_n \in \mathrm{Hom}_R(S_R, M),$$

where $f_n(x) = f(nx)$ for $x \in S_R$. We now show that Ψ is well-defined. That is, we first show that $f_n : S_R \to M$ is a homomorphism of right R-modules. Since $n \in N$ and $s \in S$, and since N is a right S-module, we have that $ns \in N$. Since $f : N_R \to M$, we have that $f(ns) \in M$. Hence f_n is a well-defined map. Now let $x, x_1, x_2 \in S_R$ and $r \in R$. Using the fact that $f : N_R \to M$ is a homomorphism of right R-modules as well as the right R-module structure of N_R and S_R, we check the axioms for f_n to be a homomorphism of right R-modules:

- $f_n(x_1 + x_2) = f(n(x_1 + x_2)) = f(nx_1 + nx_2) = f(nx_1) + f(nx_2) = f_n(x_1) + f_n(x_2)$,
- $f_n(xr) = f_n(x\phi(r)) = f(n(x\phi(r)) = f((nx)\phi(r)) = f((nx)r) = f(nx)r = f_n(x)r$.

Hence $f_n : S_R \to M$ is indeed a homomorphism of right R-modules, and so $f_n \in \mathrm{Hom}_R(S_R, M)$.

We continue by showing that $\Psi(f) \in \mathrm{Hom}_S(N, \mathrm{Hom}_R(S_R, M))$, that is that $\Psi(f) : N \to \mathrm{Hom}_R(S_R, M)$ is a homomorphism of right S-modules. Since we have shown that $f_n \in \mathrm{Hom}_R(S_R, M)$, we have that $\Psi(f)$ is a well-defined map. Now let $n, n_1, n_2 \in N$, let $s \in S$ and let $x \in S_R$. Using the fact that $f : N_R \to M$ is a homomorphism of right R-modules, as well as the right S-module structure of S and $\mathrm{Hom}_R(S_R, M)$, we check the axioms for $\Psi(f)$ to be a homomorphism of right S-modules:

- $((\Psi(f))(n_1 + n_2))(x) = (f_{n_1+n_2})(x) = f((n_1 + n_2)x) = f(n_1 x + n_2 x) = f(n_1 x) + f(n_2 x)$
  $= f_{n_1}(x) + f_{n_2}(x) = (f_{n_1} + f_{n_2})(x) = (\Psi(f)(n_1) + \Psi(f)(n_2))(x),$
- $((\Psi(f))(ns))(x) = f_{ns}(x) = f((ns)x) = f(n(sx)) = f_n(sx) = (f_n \cdot s)(x) = ((\Psi(f)(n))s)(x).$

Since the above hold for arbitrary $x \in S_R$, we obtain that

- $(\Psi(f))(n_1 + n_2) = \Psi(f)(n_1) + \Psi(f)(n_2),$
- $(\Psi(f)(ns)) = (\Psi(f)(n))s,$

which shows that $\Psi(f) : N \to \mathrm{Hom}_R(S_R, M)$ is a homomorphism of right S-modules.

Hence we have seen that Ψ is well-defined. Next we show that Ψ is bijective. We start by showing that Ψ is injective. Assume that $\Psi(f) = \Psi(g)$ for some $f, g \in \mathrm{Hom}_R(N_R, M)$. Then $f_n = g_n$ for any $n \in N$. It follows that

$$f(n) = f(n1_S) = f_n(1_S) = g_n(1_S) = g(n1_S) = g(n),$$

and so $f(n) = g(n)$ for any $n \in N$. Hence $f = g$ and so Ψ is injective.

Finally, it remains to show that Ψ is surjective. Let $u \in \mathrm{Hom}_S(N, \mathrm{Hom}_R(S_R, M))$. Then $u(n) \in \mathrm{Hom}_R(S_R, M)$ for any $n \in N$. In particular, $(u(n))(1_S) \in M$ for any $n \in N$. We define $v \in \mathrm{Hom}_R(N_R, M)$ via

$$v(n) = (u(n))(1_S).$$

We claim that this is well-defined, that is that $v : N_R \to M$ is indeed a homomorphism of right R-modules. Let $n, n_1, n_2 \in N_R$ and $r \in R$. Using the fact that $u(n) : S_R \to M$ is a homomorphism of right R-modules, that $\mathrm{Hom}_R(S_R, M)$ is a right S-module as well as the right R-module structures of N_R and S_R we check the axioms for v to be a homomorphism of right R-modules:

- $v(n_1 + n_2) = (u(n_1 + n_2))(1_S) = (u(n_1) + u(n_2))(1_S) = (u(n_1))(1_S) + (u(n_2))(1_S) = v(n_1) + v(n_2),$
- $v(nr) = v(n\phi(r))(u(n\phi(r)))(1_S) = (u(n) \cdot (\phi(r)))(1_S) = u(n)(1_S\phi(r)) = (u(n))(1_S r) = ((u(n))(1_S))r = v(n)r.$

Hence indeed $v : N_R \to M$ is well-defined. It remains to show that $\Psi(v) = u$, or that $v_n = u(n)$ for any $n \in N$. For this, let $x \in S_R$ and we have

$$v_n(x) = v(nx) = (u(nx))(1_S) = (u(n) \cdot x)(1_S) = u(n)(x1_S) = (u(n))(x),$$

as required.