Rings and modules - Problem set 3

To be solved on Tuesday 17.10

Problem 1. (After Chapter 14.3.) Let R be a ring and let $f : M \to N$ be a homomorphism of left R-modules. Show that the following hold.

(a) $f(0_M) = 0_N$.

(b) $f(m_1 - m_2) = f(m_1) - f(m_2)$ for all $m_1, m_2 \in M$.

(c) The set $\operatorname{Im} f = \{f(m) \mid m \in M\}$ is a submodule of N.

(d) The set $\operatorname{Ker} f = \{m \in M \mid f(m) = 0_N\}$ is a submodule of M.

(e) f is injective if and only if $\operatorname{Ker} f = \{0_M\}$.

(f) f is surjective if and only if $\operatorname{Im} f = N$.

Problem 2. (After Chapter 14.4.) Let F be a field and $R = M_n(F)$. Let $k \in \{1, \dots, n\}$ and e_{kk} be the matrix with 1 in position (k, k) and zero everywhere else. Show that

$$Re_{kk} = \left\{ \begin{pmatrix} 0 & \cdots & 0 & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk} & 0 & \cdots & 0 \end{pmatrix} \middle| a_{ik} \in F \right\}$$

is a simple left R-module.

Problem 3. (After Chapter 14.1.) (Exercise 14.1.3 in the book.) Let M be an additive abelian group. Show that there is only one way of making it a $\mathbb{Z}$-module.

Problem 4. (After Chapter 11.3.) Show that $\mathbb{Z}[X]$ is not a PID but is a UFD.

Problem 5. (After Chapter 14.5.) (Exercise 14.5.5 in the book.) Show that ${}_{\mathbb{Z}}\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Problem 6. (After Chapter 14.5.)

(a) (Exercise 14.5.6 in the book.) Show that every ideal of $\mathbb{Z}$ is free as a $\mathbb{Z}$-module.

(b) (Exercise 14.5.7 in the book.) Show that every principal left ideal in a unital integral domain R is free as a left R-module.

Problem 7. (After Chapter 14.2.) (Exercise 14.2.9 in the book.) Let R be the ring of all 2×2 upper triangular matrices over the field $\mathbb{Z}_2$.

(a) List all direct summands of R as a left R-module; that is, all left ideals A of the ring R such that $A \oplus B = R$ for some left ideal B of R. *(Hint: see Example 5.4 in the notes)*

(b) For each direct summand in (a) list all idempotents generating it as a left R-module.

Problem 8. (After Chapter 14.3.) Let R be a unital ring and let M be a left R-module. Let $X \subseteq M$ be a subset.

(a) Show that
$$(X) = \{r_1 x_1 + \cdots + r_m x_m \mid m > 0, r_i \in R, x_i \in X\}.$$

(b) Let $f : M \to M'$ be a left R-module homomorphism. Show that if $M = (X)$, then $\operatorname{Im} f = (f(X))$.

Problem 9. (After Chapter 14.3.) Let R be a unital ring and let $f : M \to N$ be a homomorphism of left R-modules. Let $B \subseteq M$ and let $f(B) = \{f(b) \mid b \in B\}$. Show that the following hold.

(a) Assume that f is injective. If B is linearly independent, then $f(B)$ is also linearly independent.

(b) Assume that f is surjective. If $M = (B)$, then $N = (f(B))$.

(c) Assume that f is bijective. Then B is a basis of M if and only if $f(B)$ is a basis of N.

Problem 10. (After Chapter 14.3.) Based on the second and third isomorphism theorems for rings (problems 8 and 9 in problem set 2) state the second and third isomorphism theorems for left modules over a unital ring R.

Problem 11. (After Chapter 14.4.) Let R be a unital ring.

(a) Let M be a semisimple module. Show that every nonzero submodule of M is semisimple. *(Hint: use Theorem 11.7(3) in the notes.)*

(b) (Exercise 14.4.2 in the book.) Let M be a semisimple left R-module. Suppose that N is a submodule of M with $N \neq M$. Prove that M/N is semisimple. *(Hint: use Theorem 11.7(3) in the notes to obtain that $M = N \oplus L$ for some submodule $L \subseteq M$. Show that $M/N \cong L$ using the first isomorphism theorem for modules, and then use part (a).)*

(c) (Exercise 14.4.7 in the book.) Show that ${}_RR$ is semisimple as a left R-module if and only if every left R-module is semisimple. *(Hint: For the direction ${}_RR$ semisimple implies that every left R-module is semisimple, show that if M is a left R-module, then (m) is semisimple for any $m \in M$ by using part (b).)*

Problem 12. (After Chapter 14.3.) Let R be a ring and let M be a left R-module. Let N, L, L' be submodules of M such that $N \oplus L = N \oplus L'$.

(a) (Exercise 14.2.8 in the book.) Show that it is not necessarily true that $L = L'$.

(b) Show that $(N \oplus L)/N \cong L$.

(c) (Exercise 14.3.6 in the book.) Show that $L \cong L'$.

Problem 13. (After Chapter 14.3.) (Exam August 2013, problem 5.) Let R, S be unital rings and $\phi : S \to R$ be a ring homomorphism such that $\phi(1_S) = 1_R$.

(a) What does it mean for a left R-module to be finitely generated?

(b) (i) Explain how we can view every left R-module as a left S-module. *(Hint: in other words, if ${}_RM$ is a left R-module, give to M the structure of a left S-module.)*

(ii) Assume that ϕ is surjective. Show that a finitely generated left R-module is also a finitely generated left S-module.

(c) (i) Write down Zorn's lemma.

(ii) Show that every nonzero finitely generated left R-module has a maximal submodule. *(If M is a left R-module, then a* maximal *submodule of M is a submodule $N \subseteq M$ with $N \neq M$ and such that if L is a submodule of M with $N \subseteq L$, then $N = L$ or $L = M$.)*

Problem 14. (After Chapter 14.4.) (Exam December 2016, problem 4.) Let R be a unital ring. Let M be a left R-module and let S be a simple submodule of M such that M/S is also simple. Let N be a submodule of M.

(a) Show that one of $N = 0$, $N = S$, $N = M$ and $N \cong M/S$ holds. *(Hint: use the first isomorphism theorem for modules)*

(b) Show that if $N \neq S$ and N is simple, then $N \cong M/S$ and $M \cong N \oplus S$.

(c) Give an explicit example of a unital ring R and left R-modules M, S and N as above, such that $N = S$ and $N \cong M/S$, but $M \not\cong N \times S$. *(Hint: consider $R = F[X]/(X^2)$ for some field F, $M = {}_RR$ and $N = (\overline{X})$.)*

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 15. (After Chapter 14.3.) Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A $\mathbb{C}$-*representation* of Q is a collection $M_Q = (M_v, f_\alpha)_{v \in Q_0, \alpha \in Q_1}$ such that M_v is a vector space over $\mathbb{C}$ and $f_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $\mathbb{C}$-linear map.

Now set A_2 to be the quiver $A_2 = 1 \xrightarrow{\alpha} 2$. Recall that the path algebra $\mathbb{C}A_2$ is a $\mathbb{C}$-algebra with $\mathbb{C}$-basis given by $\{e_1, e_2, \alpha\}$.

(a) Let M be a left $\mathbb{C}A_2$-module. Set $M_1 = e_1 M$, $M_2 = e_2 M$ and $f_\alpha : e_1 M \to e_2 M$ given by $f_\alpha(x) = \alpha x$. Show that $F(M) := (M_i, f_\alpha)_{i=1,2}$ is a representation of A_2.

(b) Let $M_{A_2} = (M_i, f_\alpha)_{i=1,2}$ be a representation of A_2. Define a left $\mathbb{C}A_2$-module $G(M_{A_2})$ such that $G \circ F(M) \cong M$ for any left $\mathbb{C}A_2$-module M (it also satisfies $F \circ G(M_{A_2}) \cong M_{A_2}$, if one defines a suitable notion of morphisms of representations).

Problem 16. (After Chapter 14.3.) Let R, S be unital rings and $\phi : R \to S$ be a ring homomorphism such that $\phi(1_R) = 1_S$. Let M be a right R-module and N be a right S-module.

(a) Show that N can be made into a right R-module N_R by setting $nr := n\phi(r)$ for every $n \in N$ and every $r \in R$.

(b) By (a) we may view S as a right R-module. Show that the set $\mathrm{Hom}_R(S_R, M)$ has the structure of a right S module.

(c) Provide a bijection between the sets $\mathrm{Hom}_R(N_R, M)$ and $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S_R, M))$.