Rings and modules - Problem set 2

To be solved on Tuesday 26.09

Problem 1. Let $R = (R, +, \cdot)$ be a ring. For $r, s \in R$, define

$$r \circ s := sr.$$

Show that $R^{\text{op}} = (R, +, \circ)$ is a ring, called the *opposite ring* of R.

Problem 2. (Exercise 10.2.7 in the book.) Let F be a field and R be a ring. Let $\phi : F \to R$ be a ring homomorphism. Show that $\phi = 0$ or ϕ is injective.

Problem 3. (Exercises 10.1.1 and 10.1.2 in the book) Let R be a unital ring.

(a) Show that R is a division ring if and and only if the only right (or left) ideals in R are the trivial ideals (0) and R.

(b) Assume that R is also commutative. Show that R is a field if and only if the only two-sided ideals in R are the trivial ideals $\{0\}$ and R.

Problem 4. (a) Let $n, m \in \mathbb{Z}$. Show that $(n) \subseteq (m)$ if and only if m divides n.

(b) (Exercise 10.1.4 in the book.) Find all ideals in $\mathbb{Z}/(10)$.

Problem 5. (Exercise 10.3.4 in the book.) Let R be a unital ring and $e \in R$ be an idempotent. Show that $eR \oplus (1-e)R$ is a direct sum of right ideals and that $eR \oplus (1-e)R = R$.

Problem 6. (Exercise 10.4.3 in the book.) Prove that the ideal (X^4+4) is not a prime ideal in the polynomial ring $\mathbb{Q}[X]$.

Problem 7. Let R be a commutative ring and I an ideal in R. Show that R/I is an integral domain if and only if I is a prime ideal.

Problem 8. (Third isomorphism theorem for rings.) Let R be a ring. Let I and J be ideals in R such that $I \subseteq J$. Show that the quotient ring $(R/I)/(J/I)$ is isomorphic to the quotient ring R/J. *(Hint: use the first isomorphism theorem for rings.)*

Problem 9. (Second isomorphism theorem for rings.) Let R be a ring. Let S be a subring of R and let I be an ideal in R. Show that the following hold.

(a) The sum $S + I := \{s + a \mid a \in S, a \in I\}$ is a subring of R.

(b) The intersection $S \cap I$ is an ideal in S.

(c) The quotient rings $S/(S \cap I)$ and $(S+I)/I$ are isomorphic.

(Hint: use the first isomorphism theorem for rings.)

Problem 10. (Exam December 2011, problem 3.) Let R be a unital ring. Show that every proper left ideal I in R is contained in a maximal left ideal of R.

Problem 11. (Exercise 10.2.4 in the book.) Let R be a commutative ring. Let N be the set of all nilpotent elements in R.

(a) Show that N is a nil ideal.

(b) Show that the ring R/N has no nonzero nilpotent elements.

(c) Give an example to show that if R is not commutative, then N is not necessarily an ideal.

Problem 12. Let R be a ring.

(a) For $a, b \in R$ we define

$$aRb := \{arb + nab \mid r \in R, n \in \mathbb{Z}\},$$

that is

$$aRb = \{xb \mid x \in aR\} \text{ and } aRb = \{ay \mid y \in Rb\}.$$

Show that a two-sided ideal P in R is prime, if and only if it satisfies the condition

$$aRb \subseteq P \text{ implies that } a \in P \text{ or } b \in P. \tag{1}$$

(Hint: for showing that P prime implies (1) let $aRb \subseteq P$ and consider the product of the two-sided ideals $(a)(b)$. Show that $(a)(b) \subseteq P$ by considering how a term of $(a)(b)$ looks like and using both the assumptions that $aRb \subseteq P$ and that P is a two-sided ideal.)

(b) (Exercise 10.4.6 in the book.) Let R be a ring and P be a prime (two-sided) ideal in R such that the quotient ring R/P has no nonzero nilpotent elements. Show that R/P is an integral domain.

Problem 13. (Exam December 2010, problem 2.) Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z}_2)$. Define $\psi : \mathbb{Z}_2[X] \to M_3(\mathbb{Z}_2)$ by

$$\psi(f(x)) = a_0 I_3 + a_1 A + a_2 A^2 + \cdots + a_m A^m,$$

for $f(x) = a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m \in \mathbb{Z}_2[X]$ and where $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Show that ψ is a ring homomorphism.

(b) Find the kernel $\operatorname{Ker}\psi$ of ψ, and show that the image of ψ, denoted by $\operatorname{Im}\psi$, is a subring of $M_3(\mathbb{Z}_2)$ and a field with 8 elements.

(c) Let $F = \operatorname{Im}\psi$. Why is $M_3(\mathbb{Z}_2)$ not an algebra over F, when the action of the subring F on $M_3(\mathbb{Z}_2)$ is the natural one? Find a field k such that $M_3(\mathbb{Z}_2)$ is an algebra over k, and compute the dimension of $M_3(\mathbb{Z}_2)$ as a vector space over k.

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 14. Let R be a ring.

(a) (Exercise 10.5.2 in the book.) Assume that R is commutative and let $I_1, \ldots, I_n$ be nil ideals in R. Show that $I_1 + \cdots + I_n$ is a nil ideal in R.

(b) (Exercise 10.5.1 in the book.) Let I and J be nilpotent ideals in R. Show that $I + J$ is a nilpotent ideal in R. *(Hint: a general element of $(I + J)^k$ for some $k \geq 1$ is a finite sum of terms of a specific form. Show that these terms are all zero for a big enough k.)*

Problem 15. Let R be a unital ring. Let M be the set of all non-invertible elements in R. Show that the following are equivalent.

(a) The set M is an ideal.

(b) For every $r \in R$, either r or $1 - r$ is a unit.

(Hint: to show that (b) implies (a), observe first that in any ring the product of two invertible elements is invertible, while the product of a non-invertible element with an invertible element in non-invertible.)

Rings for which these equivalent conditions hold are called *local*. Show also the following

(c) If R is a local unital ring with $0 \neq 1$, then R/M is a division ring.

(d) A commutative ring is local if and only if it has a unique maximal ideal (this holds also for non-commutative rings, but the proof is more involved.)

(e) (Exercise 10.2.19 in the book.) Show that the rings $\mathbb{Z}/(p^2)$ where p is a prime number and $F[[X]]$ where F is a field are local. *(In both cases use (d). For $\mathbb{Z}/(p^2)$ use the correspondence theorem to describe the ideals of $\mathbb{Z}/(p^2)$. For $F[[X]]$ show first that an element $f(X) = \sum_{i=0}^{\infty} a_i X^i$ of $F[[X]]$ is invertible if and only if $a_0 \neq 0$, and use this to show that any ideal of $F[[X]]$ is of the form (X^m) for some $m \geq 0$.)*