Rings and modules - Problem set 1

To be solved on Tuesday 05.09

Problem 1. Let R be a ring. Show that the polynomial ring $R[X]$ is commutative if and only if R is commutative, and is unital if and only if R is unital.

Problem 2. Let R be a ring. Show that the following are equivalent.

(a) R is an integral domain.

(b) The left cancellation property holds in R, that is if $a, b, c \in R$, $a \neq 0$ and $ab = ac$ holds, then $b = c$.

(c) The right cancellation property holds in R, that is if $a, b, c \in R$, $a \neq 0$ and $ba = ca$ holds, then $b = c$.

Problem 3. (Exercise 9.4.5(a) in the book.) Determine the idempotents, nilpotent elements and invertible elements of the following rings.

(i) $\mathbb{Z}/(4)$.

(ii) $\mathbb{Z}/(20)$.

Problem 4. Let R be a ring and $r \in R$. Let $\langle r \rangle$ be the subring generated by $\{r\}$. Show that $\langle r \rangle = X$ where

$$X = \{n_1 r + n_2 r^2 + \cdots + n_k r^k \mid n_i \in \mathbb{Z}, k > 0\}.$$

Problem 5. Let $R = M_2(\mathbb{Z})$ be the ring of 2×2 integer matrices. Recall that R is unital with $1_R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(a) Show that the subset $S \subseteq R$ defined by

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$$

is a subring of R. Show that S is not unital. Conclude that the subring of a unital ring does not need to be unital.

(b) Show that the subset $T \subseteq S$ defined by

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a0 \in \mathbb{Z} \right\}$$

is a subring of T. Show that T is unital with $1_T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Conclude that the subring of a ring which is not unital, may be unital. Conclude also that the subring of a unital ring may be unital with a different multiplicative identity.

Problem 6. Let R_i, $i = 1, 2, \ldots$ be an infinite family of unital rings.

(a) Recall that the direct product $R = \prod_{i=1}^{\infty} R_i$ is a ring under the operations

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$$
$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots),$$

for $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in R$. Is R a unital ring?

(b) Recall that the direct sum $S = \bigoplus_{i=1}^{\infty} R_i$ defined by

$$S = \{(a_1, a_2, \ldots) \in R \mid a_i = 0 \text{ for all but finitely many } i\}$$

is a subring of R. Is S a unital ring?

Problem 7. (Exercise 9.4.5(b) in the book.) Show that the set of units $U(R)$ of a unital ring R forms a multiplicative group.

Problem 8. Let R be a unital ring. Find the center of the ring $M_2(R)$ of 2×2 matrices over R.

Problem 9. (a) Let R be a unital ring with $\mathrm{char}(R) \neq 0$. Show that $\mathrm{char}(R) = \min\{n > 0 \mid n1 = 0\}$.

(b) Show that $\mathrm{char}(\mathbb{Z}/(n)) = n$.

Problem 10. Let $n \geq 2$ be an integer. Show that the ring Z_n is a field if and only if n is a prime number.

Problem 11. (Exercise 9.4.7 in the book.) Let R be a commutative ring and $r, s \in R$. If r and s are nilpotent, show that $r + s$ is also nilpotent. Show that this does not necessarily hold if R is not commutative.

Extra problems

The following problems may be a bit more challenging, in case you feel like you need something more.

Problem 12. Let $\mathbb{H} \subseteq M_2(\mathbb{C})$ be the set

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\}$$

where $\overline{a + bi} = a - bi$ indicates complex conjugation.

(a) Show that $\mathbb{H}$ is a subring of $M_2(\mathbb{C})$. Is $\mathbb{H}$ commutative? Is $\mathbb{H}$ unital?

(b) Is $\mathbb{H}$ a division ring? Is $\mathbb{H}$ a field?

(c) Find the center $Z(\mathbb{H})$ of $\mathbb{H}$.

Problem 13. (Exercise 9.4.5(c) in the book.) Let $n \geq 1$ be an integer. Prove that an element $\overline{x} \in \mathbb{Z}/(n)$ is invertible if and only if $\gcd(x, n) = 1$. Show also that if $\gcd(x, n) = 1$, then $x^{\phi(n)} \equiv 1 \pmod{n}$ where $\phi(n)$ is Euler's function.