All rings are unital, that is they have a multiplicative identity. Motivate all your answers!

Problem 1. Let $A = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}$ be a matrix over $\mathbb{Q}$.

(a) Find the Smith normal form of the matrix

$$A - XI_3 = \begin{pmatrix} 2-X & 3 & 6 \\ 0 & -X & -4 \\ 0 & 1 & 4-X \end{pmatrix}$$

over $\mathbb{Q}[X]$.

(b) Compute the rational canonical form of A over $\mathbb{Q}$.

(c) Compute the Jordan canonical form of A over $\mathbb{Q}$.

Solution.

(a) We have

$$\begin{pmatrix} 2-X & 3 & 6 \\ 0 & -X & -4 \\ 0 & 1 & 4-X \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 3 & 2-X & 6 \\ -X & 0 & -4 \\ 1 & 0 & 4-X \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 4-X \\ -X & 0 & -4 \\ 3 & 2-X & 6 \end{pmatrix} \xrightarrow{C_3 \to C_3 - (4-X)C_1}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ -X & 0 & -X^2+4X-4 \\ 3 & 2-X & 3X-6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + XR_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -X^2+4X-4 \\ 0 & 2-X & 3X-6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2-X & 3X-6 \\ 0 & 0 & -X^2+4X-4 \end{pmatrix}$$
$$\xrightarrow{C_3 \to C_3 + 3(C_2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2-X & 0 \\ 0 & 0 & -X^2+4X-4 \end{pmatrix} \xrightarrow[R_3 \to -R_3]{R_2 \to -R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & X-2 & 0 \\ 0 & 0 & (X-2)^2 \end{pmatrix},$$

and $1 \mid (X-2) \mid (X-2)^2$. Hence the Smith normal form of $A - XI_3$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & X-2 & 0 \\ 0 & 0 & (X-2)^2 \end{pmatrix}$.

(b) By part (a) we have that the non-unit monic invariant factors of $A - XI_3$ are $X-2$ and $(X-2)^2 = X^2 - 4X + 4$. Hence the rational canonical form of A is

$$\begin{pmatrix} C_{X-2} & \\ & C_{X^2-4X+4} \end{pmatrix} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 0 & -4 \\ 0 & 1 & 4 \end{array}\right).$$

(c) By part (a) we have that the elementary divisors of A are $X-2$ and $(X-2)^2$. It follows that the Jordan canonical form of A is

$$\left(\begin{array}{c|c} J_{1,X-2} & 0 \quad 0 \\ \hline 0 & \\ 0 & J_{2,X-2} \end{array}\right) = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & 0 \\ 0 & 1 & 2 \end{array}\right).$$

Problem 2. Let K be a field and let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a,b,c,d,e,f \in K \right\}, \quad I = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & f \end{pmatrix} \mid c,e,f \in K \right\}.$$

(a) Show that R is a subring of the ring $M_{3\times 3}(K)$ of 3×3 matrices over K and that it contains the multiplicative identity of $M_{3\times 3}(K)$.

(b) Show that I is a two-sided ideal of R and find a subring S of the ring $M_{2\times 2}(K)$ of 2×2 matrices over K which is isomorphic to R/I.

(c) Find a nonzero nilpotent two-sided ideal and a maximal two-sided ideal in the ring S. Is S a semisimple ring? Is S a noetherian ring? Is S an artinian ring?

Solution.

(a) For matrices

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} \in R$$

we have

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} + \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' & c+c' \\ 0 & d+d' & e+e' \\ 0 & 0 & f+f' \end{pmatrix} \in R,$$

and

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} = \begin{pmatrix} aa' & ab'+bd' & ac'+ce'+ff' \\ 0 & dd' & de'+ef' \\ 0 & 0 & ff' \end{pmatrix} \in R,$$

and also

$$1_{M_{3\times 3}(K)} = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R.$$

Hence R is a subring of $M_{3\times 3}(K)$ and the multiplicative identity of $M_{3\times 3}(K)$ is in R.

(b) Clearly $I \neq \varnothing$. For matrices

$$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} 0 & 0 & x' \\ 0 & 0 & y' \\ 0 & 0 & z' \end{pmatrix} \in I_2 \text{ and } \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in R$$

we have

$$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{pmatrix} + \begin{pmatrix} 0 & 0 & x' \\ 0 & 0 & y' \\ 0 & 0 & z' \end{pmatrix} = \begin{pmatrix} 0 & 0 & x+x' \\ 0 & 0 & y+y' \\ 0 & 0 & z+z' \end{pmatrix} \in I,$$

and

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & ax+by+cz \\ 0 & 0 & de+ez \\ 0 & 0 & fz \end{pmatrix} \in I,$$

and also

$$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & xf \\ 0 & 0 & yf \\ 0 & 0 & zf \end{pmatrix} \in I.$$

Hence I is a two-sided ideal of R.

Define $\phi : R \to M_{2\times 2}(K)$ via

$$\phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

We claim that ϕ is a ring homomorphism. Indeed for matrices

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} \in R$$

we have

$$\begin{aligned}
\phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} + \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} a+a' & b+b' & c+c' \\ 0 & d+d' & e+e' \\ 0 & 0 & f+f' \end{pmatrix}\right) \\
&= \begin{pmatrix} a+a' & b+b' \\ 0 & d+d' \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \\
&= \phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) + \phi\left(\begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}\right),
\end{aligned}$$

and

$$\begin{aligned}
\phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} aa' & ab'+bd' & ac+be'+cf' \\ 0 & dd' & de'+ef' \\ 0 & 0 & ff' \end{pmatrix}\right) \\
&= \begin{pmatrix} aa' & ab'+bd' \\ 0 & dd' \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \\
&= \phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) \phi\left(\begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}\right).
\end{aligned}$$

We claim that $\ker\phi = I$. Indeed, for a matrix $\left(\begin{smallmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & f \end{smallmatrix}\right) \in I$ we have

$$\phi\left(\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so $\left(\begin{smallmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & f \end{smallmatrix}\right) \in \ker\phi$, showing $I \subseteq \ker\phi$. On the other hand for $\left(\begin{smallmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{smallmatrix}\right) \in \ker\phi$ we have

$$\phi\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies a = b = d = 0,$$

and so $\left(\begin{smallmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & c \\ 0 & 0 & e \\ 0 & 0 & f \end{smallmatrix}\right) \in I$. This shows that $\ker\phi \subseteq I$ and so we conclude that $\ker\phi = I$.
Next it is clear from the definition of ϕ that

$$\operatorname{Im}\phi = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in K\right\} =: S.$$

Hence by the first isomorphism theorem for rings we conclude that

$$R/I = R/\ker\phi \cong \operatorname{Im}\phi = S.$$

(c) Consider the two-sided ideal N generated by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in S$. Then its elements are sums of elements of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} 0 & ad' \\ 0 & 0 \end{pmatrix},$$

and so

$$N = \left\{\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in K\right\}.$$

Since

$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we conclude that $N^2 = 0$ and so N is a nilpotent two-sided ideals.

Now consider the two-sided ideal M generated by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$. Then we claim that

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in K \right\}.$$

Indeed, for every $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in S$ with $a, b \in K$ we have

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in M$$

showing one inclusion. For the other inclusion notice that all matrices in M are sums of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' \\ 0 & 0 \end{pmatrix}.$$

and so they belong to $\{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in K\}$ This shows the claim. Now we claim that M is a maximal ideal. Let $M \subseteq M' \subseteq S$ where M' is another two-sided ideal of S. Assume that $M \neq M'$ and we show that $M' = S$. Indeed, since $M \neq M'$, there exists at least one matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) \in M'$ with $d \neq 0$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\in M \subseteq M'} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}}_{\in M'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M'.$$

Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in M'$, we conclude that $M' = S$ as required and so M is a maximal two-sided ideal.

S is noetherian and artinian since it is a finite-dimensional K-algebra. It is not semisimple since it has a nonzero nilpotent ideal.

Problem 3. Let K be a field and let $K[X]$ be the polynomial ring over K.

(a) Consider $K[X]$ as a $K[X]$-module. Is it a free module? Is it a simple module?

(b) Consider the submodule M of $K[X]$ generated by $\{X\}$, that is

$$M = (X) = \{f(X)X \mid f(X) \in K[X]\}.$$

Let N be the quotient module $N = K[X]/M$. Is N a free module? Is N a simple module?

Solution.

(a) $K[X]$ is a free $K[X]$-module with basis $\{1\}$. Indeed for any $p(X) \in K[X]$ we have that $p(X) = p(X) \cdot 1$, which shows that $\{1\}$ is linearly independent and generates $K[X]$.

$K[X]$ is not a simple module as (X) is a submodule of $K[X]$ and $(X) \neq (0)$ since $X \in (X)$ while $(X) \neq K[X]$ since $1 \notin (X)$.

(b) Let $p(X) + (X) \in N = K[X]/(X)$. Notice that for $X \in K[X]$ we have

$$X(p(X) + (X)) = Xp(X) + (X) = 0 + (X).$$

Hence no element of N is linearly independent. Therefore N is not free.

Now let $L \subseteq N$ be a submodule. Assume that $L \neq 0$ and so there exists $p(X) + (X) \in L$ with $p(X) + (X) \neq 0 + (X)$. If $p(X) = a_0 + a_1X + \cdots + a_nX^n$, then

$$p(X) + (X) = a_0 + a_1X + \cdots + a_nX^n + (X) = a_0 + (X),$$

and since $a_0 + (X) \neq 0 + (X)$, we have that $a_0 \neq 0$. Hence

$$(a_0^{-1} + (X))(a_0 + (X)) = 1 + (X) \in L.$$

Then for every element $q(X) + (X) \in N$ we have

$$q(X)(1 + (X)) = q(X) + (X) \in L,$$

and so $N \subseteq L$. We conclude that $L = N$ and so N is indeed a simple $K[X]$-module.

Problem 4. Let R be a principal ideal domain (PID).

(a) State the structure theorem for finitely generated modules over R.

(b) Recall that the *torsion submodule* of M is the submodule

$$\operatorname{Tor} M = \{m \in M \mid \text{ there exists nonzero } r \in R \text{ with } rm = 0\}.$$

Show that if M is finitely generated, then the quotient module $\frac{M}{\operatorname{Tor} M}$ is free.

(c) Show that part (b) does not necessarily hold if M is not finitely generated by considering $\mathbb{Q}$ as a $\mathbb{Z}$-module. That is, show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module and that $\frac{\mathbb{Q}}{\operatorname{Tor} \mathbb{Q}}$ is not a free module.

Solution.

(a) **Theorem.** Let M be a finitely generated module over a PID R. Then there exist an integer $s \geq 0$ and nonzero nonunits $a_1, \dots, a_u \in R$ with $a_1 \mid a_2 \mid \cdots \mid a_u$ such that

$$M \cong R^s \oplus \frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_u)}.$$

Moreover this decomposition is unique up to multiplication of the a_i's by units.

(b) By the structure theorem for finitely generated modules over R we know that there exist an integer $s \geq 0$ and nonzero nonunits $a_1, \dots, a_u \in R$ with $a_1 \mid a_2 \mid \cdots \mid a_u$ such that

$$M \cong R^s \oplus \frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_u)}.$$

Moreover, we know that $\operatorname{Tor} M \cong \frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_u)}$. Hence

$$\frac{M}{\operatorname{Tor} M} \cong \frac{R^s \oplus \frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_u)}}{\frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_u)}} \cong R^s,$$

and so $\frac{M}{\operatorname{Tor} M}$ is free.

(c) First we show that $\mathbb{Q}$ is not finitely-generated as a $\mathbb{Z}$-module. Indeed, assume to a contradiction that $\mathbb{Q}$ is generated by $\{\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\}$ as a $\mathbb{Z}$-module, where $a_i, b_i \in \mathbb{Z}$. Then for every $q \in \mathbb{Q}$ there exist $k_1, \dots, k_n \in \mathbb{Z}$ such that

$$q = k_1 \frac{a_1}{b_1} + \cdots + k_n \frac{a_b}{b_n} = \frac{x}{l}$$

for some $x \in \mathbb{Z}$. But this is clearly impossible, for example by picking $q = \frac{1}{p}$ where p is a prime not dividing l.

Now notice that $\operatorname{Tor} \mathbb{Q} = (0)$ since for every nonzero $k \in \mathbb{Z}$ and every $\frac{p}{q} \in \mathbb{Q}$ we have

$$k\frac{p}{q} = 0 \implies \frac{p}{q} = 0.$$

Hence $\frac{\mathbb{Q}}{\operatorname{Tor} \mathbb{Q}} = \frac{\mathbb{Q}}{\operatorname{Tor} \mathbb{Q}} = \mathbb{Q}$ and we need to show that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module. Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, where $p, q, r, s \in \mathbb{Z}$. Then $qr, -ps \in \mathbb{Z}$ and so

$$qr\frac{p}{q} + (-ps)\frac{r}{s} = rp - pr = 0.$$

Hence the elements $\frac{p}{q}$ and $\frac{r}{s}$ are not linearly independent. Hence any potential $\mathbb{Z}$-basis of $\mathbb{Q}$ would have to contain exactly one element. But then $\mathbb{Q}$ would be finitely generated, which we have shown already that is not true. Hence $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module.