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Examination paper for **MA3201 Rings and modules**

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Date Signature

Problem 1

a) Find the Smith normal form of the matrix $\begin{pmatrix} X-7 & -2 & 12 \\ -4 & X-5 & 12 \\ -2 & 1 & X+3 \end{pmatrix}$ over $\mathbb{C}[X]$.

b) Find the rational normal form and the Jordan canonical form of the matrix $\begin{pmatrix} 7 & 2 & -12 \\ 4 & 5 & -12 \\ 2 & 1 & -3 \end{pmatrix}$ over $\mathbb{C}$.

Problem 2 Let

$$\Lambda_1 = \{ \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Q} \}$$

and

$$\Lambda_2 = \{ \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \}.$$

a) Prove that Λ_1 and Λ_2 are subrings of the full matrix ring over $\mathbb{Q}$ and $\mathbb{Z}_2$ respectively, and that Λ_1 has no nilpotent elements.

b) Prove that the map $\Phi_1 \colon \Lambda_1 \to \mathbb{Q}$ given by

$$\Phi_1(\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}) = a + b + c + d$$

is a ring homomorphism, and that $\Phi_2 \colon \Lambda_2 \to \mathbb{Z}_2$ given by

$$\Phi_2(\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}) = a + b + c + d$$

is a ring homomorphism.

c) Find a $\mathbb{Q}$ basis for the kernel of Φ_1 and a $\mathbb{Z}_2$ basis for the kernel of Φ_2. Show that the kernel of Φ_2 is nilpotent.

d) Prove that $\Psi_1 \colon \Lambda_1 \to \mathbb{Q} \times \mathbb{Q}$ given by

$$\Psi_1(\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} = (a + b - c - d, a - b + c - d)$$

is a surjective ringhomomorphism, and find a ring isomorphism

$$\Psi_2 \colon \Lambda_1 \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$$

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Problem 3 Let i be a complex number such that $i^2 = -1$ and consider the abelian subgroup $A_2 = \{e^{\frac{l2\pi i}{2^k}} \mid k, l \in \mathbb{N}\}$ of the nonzero complex numbers under multiplication.

a) Prove that for each natural number k, the group A_2 containes one and only one subgroup of order 2^k, and that each cylic subgroup of A_2 has order 2^k for some natural number k.

b) Prove that A_2 is not noetherian as an abelian group. Prove that each proper subgroup of A_2 is finite, and conclude that A_2 is artinian.

c) Prove that every nonzero group homomorphism $\phi \colon A_2 \to A_2$ is surjective and show that the ring of abelian group endomorphisms of A_2 is an integral domain. (It is also commutative, but you need not prove that)

Problem 4

Prove that if Λ is a ring and M is an artin Λ-module, then every injective Λ-homomorphism $f \colon M \to M$ is an isomorphism.