MA3201 Exam - Fall 2019 - Possible solutions / sketch

(1a) • $R[x]$ is a ring, hence an <u>abelian group</u> under addition.

• For $r_1, r_2 \in R$, $p(x), q(x) \in R[x]$,

$$\left.\begin{array}{l} \bullet\ r_1(p(x)+q(x)) = r_1 p(x) + r_1 q(x) \\ \bullet\ (r_1+r_2)p(x) = r_1 p(x) + r_2 p(x) \\ \bullet\ (r_1 r_2)p(x) = r_1(r_2 p(x)) \\ \bullet\ 1 p(x) = p(x) \end{array}\right\} \text{from operations in } R[x].$$

Thus $R[x]$ is an R-module via scalar multiplication from R.

To show $R[x]$ is a free R-module, note that $\{1, x, x^2, \dots\}$ is both lin ind. and generates (it is a basis) over R,

so $R[x] \underset{\text{as } R\text{-modules}}{\cong} \bigoplus_{i=0}^{\infty} Rx^i$.

(1b) Define $\varphi : A+B \longrightarrow B/(A\cap B)$

$a+b \longmapsto b + A\cap B$.

• It is onto by definition.

• Check it is <u>well-defined</u> and a R-<u>homomorphism</u>.

• Show $\ker\varphi = A$:

If $a \in A$, then $\varphi(a) = \varphi(a+0) = 0 + A\cap B$.
$\Rightarrow a \in \ker\varphi$.

If $a+b \in \ker\varphi$, then $a+b \in A \cap B$
$\Rightarrow b \in A$.

• Fundamental isomorphism theorem $\Rightarrow \dfrac{A+B}{A} \cong \operatorname{im}\varphi = B/(A\cap B)$

(1c) Let J = max ideal of R. If A, B are ideals such that $AB \subseteq J$, and $A \nsubseteq J$, then maximality $\Rightarrow$ $A + J = R$.
Thus we may find $a \in A$, $x \in J$ such that $a + x = 1$.
For any $b \in B$, $\underbrace{ab}_{\in AB \in J} + \underbrace{xb}_{\in J} = 1b = b$,
hence $b \in J$ $\forall b \in B$ $\Rightarrow$ $B \subseteq J$.

Thus J is prime. ∎

(2a) F is a field, hence noetherian, so Hilbert's basis theorem $\Rightarrow$ $F[x]$ is noetherian. (Alternative: $F[x]$ is a PID $\Rightarrow$ noeth.)
$F[x]$ is not artinian, as there exists a properly descending chain of ideals: $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$

(2b) To show I is an ideal, check it is nonempty, closed under subtraction, and for any $A \in R$, both $IA, AI \in R$ (show this).
Also, show $I^3 = 0$, hence R is not semisimple because it has a nonzero nilpotent ideal (by Wedderburn-Artin).
On the other hand, $R/I = \begin{bmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix} \cong F \times F \times F$

$\Rightarrow$ R/I is semi-simple by W-A.

(3a) Let $\varphi: M \to N$ be an isomorphism. If $r \in \operatorname{Ann} M$, then
For $n \in N$, there exists $m \in M$ with $\varphi(m) = n$,
hence $r \cdot n = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$ $\Rightarrow$ $\operatorname{Ann} M \subseteq \operatorname{Ann} N$. A symmetrical argument shows $\operatorname{Ann} N \subseteq \operatorname{Ann} M$.
For converse, note that the $\mathbb{Z}$-modules $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are not isomorphic but both have annihilator $= 2\mathbb{Z}$.

(3b) • If M is simple, then for any nonzero $x \in M$, $Rx \subseteq M$ is a nonzero submodule, hence $Rx = M$.

• Assume $M = Rx$ for any nonzero $x \in M$ such that $\mathrm{Ann}(Rx)$ is prime. As M is f.g. torsion over a PID, we have

$$M \cong R/(p_1^{\alpha_1}) \times \cdots \times R/(p_t^{\alpha_t}),$$ some primes p_i, integers α_i.

Write $M = Rx_1 \oplus \cdots \oplus Rx_t$, where $\mathrm{Ann}(Rx_i) = (p_i^{\alpha_i})$.

Note that $R(p_1^{\alpha_1 - 1} x_1) \subseteq M$ with

$\mathrm{Ann}(R p_1^{\alpha_1-1} x) = p_1$, which is prime.

Thus $M = R p_1^{\alpha_1} x_1 \Rightarrow M \cong R/(p_1)$.

As prime ideals are maximal in a PID, we obtain that $R/(p_1)$ has no ~~nonzero and~~ nontrivial nontrivial left ideals (as a ring) hence M has no nontrivial (left) R-submodules $\Rightarrow M$ is simple. ∎

(4a)
$$\begin{bmatrix} 3 & -4 & -6 \\ 3 & 3 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_2 - R_3 \to R_2]{} \begin{bmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{} \begin{bmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{bmatrix}$$

$$\xrightarrow[C_1 \leftrightarrow C_2]{} \begin{bmatrix} -4 & 3 & -6 \\ 1 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 3 & -6 \\ 6 & 0 & 6 \end{bmatrix} \xrightarrow[\substack{R_2 + 4R_1 \to R_2 \\ R_3 - 6R_1 \to R_3}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow[C_3 + 2C_2 \to C_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$ note $1 \mid 3 \mid 6$ in $\mathbb{Z}$.

(4b) • Assume first B is similar to C. Then RCF of B and C is the same, so B and C have same invariant factors, and in particular, the same minimal polynomial.

• Now assume $m_B(x) = m_C(x)$.

<u>case 1</u>: $m_B(x) = m_C(x)$ is quadratic, so $= x^2+\alpha x+\beta$.

$\Rightarrow$ RCF of B and C is $\begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix}$,

hence B and C are similar.

<u>case 2</u>: $m_B(x) = m_C(x)$ is linear, so $= x-\gamma$.

Since $c_A(x) =$ product of invariant factors, is degree 2, and all invariant factors divide $m_A(x)$, must have $c_A(x) = (x-\gamma)^2$.

$\Rightarrow$ RCF of A is $\begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$,

and similarly for RCF of B, hence B is similar to A.

(4c) inv. factors: $x-2$, $\underbrace{(x-2)(x-5)}_{=x^2-7x+10}$, $\underbrace{(x-2)(x-5)^2}_{=x^3-12x^2+45x-50}$

elem. div: $x-2$, $x-2$, $x-5$, $x-2$, $(x-5)^2$.

RCF: $\begin{bmatrix} 2 & & & & & \\ & 0 & -10 & & & \\ & 1 & 7 & & & \\ & & & 0 & 0 & 50 \\ & & & 1 & 0 & -45 \\ & & & 0 & 1 & 12 \end{bmatrix}$

JCF: $\begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 5 & & & \\ & & & 2 & & \\ & & & & 5 & 1 \\ & & & & 0 & 5 \end{bmatrix}$

Since JCF is not diagonal, the matrix D is not diagonalizable.