Notes: All rings have unity (all rings have a 1). $\mathbb{Z}$ denotes the ring of integers and $\mathbb{R}$ denotes the field of real numbers. All answers must be justified. Each sub-problem is worth the same for grading.

Problem 1. Let R be a ring.

(1.a) Prove that the polynomial ring $R[x]$ is an R-module, and show that it is free as an R-module.

(1.b) If A and B are R-submodules of an R-module M, show that there is an R-module isomorphism: $(A+B)/A \cong B/(A \cap B)$.

(1.c) Prove that if J is a maximal ideal of R, then J is a prime ideal.

Problem 2. Let F be any field.

(2.a) Prove that the polynomial ring $F[x]$ is noetherian, but that it is not artinian.

(2.b) Consider the ring $R = \begin{bmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$ and show that $I = \begin{bmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix} \subseteq R$ is an ideal. Determine whether R and R/I are semisimple rings (and justify).

Problem 3. Let R be a ring.

(3.a) The annihilator of an R-module M is $\mathrm{Ann}(M) = \{r \in R \mid rM = 0\}$. If M and N are R-modules, prove that if $M \cong N$, then $\mathrm{Ann}(M) = \mathrm{Ann}(N)$. Give an example to show the converse of this statement is false.

(3.b) Let R be a principal ideal domain and let M be a finitely generated torsion R-module. Prove that M is irreducible (or simple) if and only if $M = Rx$ for any nonzero element $x \in M$ such that $\mathrm{Ann}(Rx)$ is a prime ideal.

Problem 4.

(4.a) Determine the Smith normal form of the matrix $A = \begin{bmatrix} 3 & -4 & -6 \\ 3 & 3 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ over $\mathbb{Z}$.

(4.b) Let B and C be 2×2 matrices over $\mathbb{R}$. Prove that B is similar to C if and only if their minimal polynomials are equal, that is, $m_B(x) = m_C(x)$.

(4.c) Let D be a 6×6 matrix with entries in $\mathbb{R}$ and with characteristic polynomial $c_D(x) = (x-2)^3(x-5)^3$ and minimal polynomial $m_D(x) = (x-2)(x-5)^2$. Determine the rational canonical form of D, the Jordan canonical form of D, and determine whether D is similar to a diagonal matrix.