MA 3201 Fall 2018 Final Exam Solutions

(1a) Let $f: M \to N$ be an R-module homomorphism.

Claim: $\ker(f) = \{x \in M \mid f(x) = 0\}$ is an R-submodule.

Pf: • As $0 \in \ker f$ ($f(0) = 0$), $\ker f \neq \emptyset$.

• Let $x, y \in \ker f$.

Then $f(x-y) = f(x) - f(y)$, since f is hom.

$= 0 - 0$, since $x, y \in \ker f$

$= 0$

$\Rightarrow x - y \in \ker f$.

• Let $r \in R$, $x \in \ker f$.

Then $f(rx) = r f(x)$, since f homomorphism

$= r \cdot 0$, since $x \in \ker f$

$= 0$. //

(1b) Assume M, N are simple and $f: M \to N$ is nonzero homom.

Claim: f is an isomorphism.

Pf: • As $\ker f$ is a submodule of M, it must be either 0 or M as M is simple.

But $\ker f \neq M$ since this would imply $f = 0$.

$\Rightarrow \ker f = 0 \Rightarrow f$ is one-to-one.

• As $\text{image}(f) \subseteq N$ is a submodule, it is either 0 or N since N is simple.

But $\text{image}(f) \neq 0$ since f is nonzero

$\Rightarrow \text{image}(f) = N \Rightarrow f$ is onto.

Thus f is 1-1 and onto R-homomorphism, hence an isomorphism.

(1c) R = nonzero, commutative ring, with 1.
M = maximal ideal of R.
Claim: M is prime ideal.

Pf #1 (doesn't need commutative)
Let $A, B \subseteq R$ be ideals such that $AB \subseteq M$.
Suppose $A \not\subseteq M$. Then $M + A = R$
($M \subsetneq M+A$, and maximality of $M \Rightarrow M+A = R$).
Write $1 = m + a$, $m \in M$, $a \in A$.
Let $b \in B$.
Then $b = \underbrace{mb}_{\in M} + \underbrace{ab}_{\in AB \subseteq M} \Rightarrow b \in M$.
Thus $B \subseteq M$ $\therefore$ Definition of prime implies M is prime ideal

Pf #2: (uses commutative)
As M is prime, we obtain R/M is a field.
Every field is an integral domain,
thus R/M is a domain $\Rightarrow M$ is prime.

OR: M prime $\Rightarrow R/M$ field $\Rightarrow R/M$ domain,
Let $a, b \in R$ such that $ab \in M$.
Suppose $a \notin M$.
Then $\underset{(a+M)(b+M)}{\underset{\shortparallel}{ab+M}} = 0 + M$
As R/M domain, $b + M \neq 0 + M$, we must
have $a + M = 0 + M \Rightarrow a \in M$.
Thus M is prime ideal.

(2a) Define $\varphi: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ by $\varphi(n) = (n+2\mathbb{Z}, n+3\mathbb{Z})$.

- Ring homomorphism:

$$\varphi(m+n) = (m+n+2\mathbb{Z}, m+n+3\mathbb{Z})$$
$$= (m+2\mathbb{Z}, m+3\mathbb{Z}) + (n+2\mathbb{Z}, n+3\mathbb{Z})$$
$$= \varphi(m) + \varphi(n)$$
$$\varphi(mn) = (mn+2\mathbb{Z}, mn+3\mathbb{Z})$$
$$= (m+2\mathbb{Z}, m+3\mathbb{Z})(n+2\mathbb{Z}, n+3\mathbb{Z})$$
$$= \varphi(m)\varphi(n).$$

- onto:

$$\left.\begin{aligned}
\varphi(0) &= (0+2\mathbb{Z}, 0+3\mathbb{Z}) \\
\varphi(1) &= (1+2\mathbb{Z}, 1+3\mathbb{Z}) \\
\varphi(2) &= (0+2\mathbb{Z}, 2+3\mathbb{Z}) \\
\varphi(3) &= (1+2\mathbb{Z}, 0+3\mathbb{Z}) \\
\varphi(4) &= (0+2\mathbb{Z}, 1+3\mathbb{Z}) \\
\varphi(5) &= (1+2\mathbb{Z}, 2+3\mathbb{Z})
\end{aligned}\right\} \text{6 elmts, order of } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \text{ is 6.}$$

- $\ker\varphi = 6\mathbb{Z}$.

$\subseteq$: Let $\varphi(n) = 0$. Then $\left.\begin{aligned} n \bmod 3 = 0 \\ n \bmod 2 = 0 \end{aligned}\right\} \Rightarrow n \in 6\mathbb{Z}$.

$\supseteq$: $\varphi(6n) = (6n+2\mathbb{Z}, 6n+3\mathbb{Z}) = (0+2\mathbb{Z}, 0+3\mathbb{Z})$.

Thus 1st iso / fundamental theorem for ring hom:

$$\Rightarrow \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \text{ (as rings).}$$

(2b) $\mathbb{Z}/6\mathbb{Z}$ has a basis over $\mathbb{Z}/6\mathbb{Z}$, namely $\{1+6\mathbb{Z}\}$.

$\mathbb{Z}/6\mathbb{Z}$ does <u>not</u> have a basis over $\mathbb{Z}$.

↳ To see this, ~~let~~ Suppose $a+6\mathbb{Z}$ ~~be~~ is any element in $\mathbb{Z}/6\mathbb{Z}$.
~~in a basis ...~~.

But then $\underset{\mathbb{Z}}{\underset{\uparrow}{6}} \cdot (a+6\mathbb{Z}) = 0+6\mathbb{Z}$,

so any nonempty set in $\mathbb{Z}/6\mathbb{Z}$ cannot be linearly independent.
This forces the only possible basis to be $\emptyset$, ~~then~~ but $\mathbb{Z}/6\mathbb{Z} \neq 0$.
Thus $\mathbb{Z}/6\mathbb{Z}$ cannot have ~~a~~ basis over $\mathbb{Z}$.

(3a) **$\mathbb{Z}$ is noetherian:**
Suppose $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals.
As $\mathbb{Z}$ is a PID, we have $I_i = (n_i)$ for some integers n_i.
But $(n_i) \subseteq (n_j) \Rightarrow n_j \mid n_i$.
As integers cannot be infinitely divisible, this sequence must stabilize.

$\mathbb{Z}$ is not artinian
for example,
$$(2) \supsetneq (2^2) \supsetneq (2^3) \supsetneq (2^4) \supsetneq \cdots$$
is an infinite (non-stabilizing) descending chain of ideals in $\mathbb{Z}$.

(3b) Claim: $\mathbb{R}[x]$ is noetherian.
Pf #1: $\mathbb{R}$ is a field, hence noetherian (only ideals are 0 and $\mathbb{R}$).
So by Hilbert basis theorem $\mathbb{R}[x]$ is noetherian.

Pf #2: $\mathbb{R}[x]$ is a PID, so if
$$I_1 \subseteq I_2 \subseteq \cdots$$
is an ascending chain of ideals, then $\bigcup_{i \geq 1} I_i$ is an ideal (show),
so $\bigcup_{i \geq 1} I_i = (x)$ some . $x \in I_j$ some j.
$\Rightarrow \bigcup I_i \subseteq I_j \Rightarrow I_j = I_{j+1} = I_{j+2} = \cdots$,
hence $\mathbb{R}[x]$ noetherian.

(3c) $R_1 = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$, $R_2 = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix}$.

Claim: R_1 is not semisimple

Pf: $I = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$ is an ideal:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ar \\ 0 & 0 \end{bmatrix} \in I$$

$$\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & rc \\ 0 & 0 \end{bmatrix} \in I$$

$$\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & r' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r+r' \\ 0 & 0 \end{bmatrix} \in I$$

But I is nilpotent:

$$\underset{\substack{\uparrow \\ I}}{\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}} \underset{\substack{\uparrow \\ I}}{\begin{bmatrix} 0 & r' \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad \Rightarrow I^2 = 0.$$

As $I \neq 0$ is nilpotent, R_1 cannot be semisimple, by Wedderburn-Artin theorem //

Claim: R_2 is semisimple.

Pf: $R_2 \cong \mathbb{R} \times \mathbb{R}$, a finite product of fields.

Wedderburn-Artin $\Rightarrow R_2$ semisimple.

(4a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 4 & 1 \end{bmatrix}$.

Find Smith normal form: over $\mathbb{Z}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 4 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{C_2 - 2C_1 \\ C_3 - 3C_1}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow[C_3 - 2C_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow[-C_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Note $1 \mid 2$, $2 \mid 6$.

(4b) companion matrix for f_1: $\begin{bmatrix} -1 \end{bmatrix}$

companion matrix for f_2: $\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$

$f_2 = x^2 + 4x + 3$

companion matrix for f_3: $\begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -15 \\ 0 & 1 & -7 \end{bmatrix}$

$$f_3 = (x^2+4x+3)(x+3)$$
$$= x^3 + 4x^2 + 3x + 3x^2 + 12x + 9$$
$$= x^3 + 7x^2 + 15x + 9$$

Rational canonical form

$$\begin{bmatrix} -1 & & & & & \\ & 0 & -3 & & & \\ & 1 & -4 & & & \\ & & & 0 & 0 & -9 \\ & & & 1 & 0 & -15 \\ & & & 0 & 1 & -7 \end{bmatrix}$$