Norges teknisk–
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Exam in course MA3201 Rings and modules

English
Wednesday November 30, 2005
Time: 09.00-13.00

Permitted aids: none

Grades: 21.12.2005.

Problem 1 Let q be a fixed non-zero element in $\mathbb{C}$, the set of complex numbers. Define the subset R_q of the ring of 4×4-matrices over $\mathbb{C}$ by

$$R_q = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & -qb & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

a) Show that R_q is a ring.

b) For which q in $\mathbb{C}$ is R_q a commutative ring?

c) For a given element α in $\mathbb{C}$ define the subset

$$I_\alpha = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ \alpha b & 0 & 0 & 0 \\ d & \alpha b & -qb & 0 \end{pmatrix} \mid b, d \in \mathbb{C} \right\}$$

of R_q. Show that I_α is a left ideal in R_q for all α in $\mathbb{C}$.

d) Show that each of the left ideals I_α is generated by one element as a left ideal. Show that $I_\alpha \simeq R/I_{\alpha q}$ as left R-modules.

Problem 2 Let $\mathbb{Q}$ be the field of rational numbers, and let a and b in $\mathbb{Q}$ be different elements. Find all possible rational canonical forms for 4×4-matrices over $\mathbb{Q}$ having

$$(x+a)^2(x+b)$$

as a minimal polynomial.

Problem 3 Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the polynomial ring over $\mathbb{C}$ in one variable x. Let $\alpha \in \mathbb{C}$ be a complex number.

a) Show that the map $\varphi_\alpha \colon \mathbb{C}[x] \to \mathbb{C}$ defined by $\varphi_\alpha(f(x)) = f(\alpha)$ is a surjective ring homomorphism, and use this to show that the ideal generated by $x - \alpha$ is a maximal ideal in $\mathbb{C}[x]$.

b) For which $n \geq 1$ is the ring

$$\begin{pmatrix} \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \\ \frac{\mathbb{C}[x]}{((x-\alpha)^n)} & \frac{\mathbb{C}[x]}{((x-\alpha)^n)} \end{pmatrix}$$

semisimple?

Problem 4 Let R be a ring, and let M be a Noetherian left R-module. Show that any surjective R-homomorphism $f \colon M \to M$ is an isomorphism. (Hint: Consider the chain $\operatorname{Ker} f \subseteq \operatorname{Ker}(f^2) \subseteq \operatorname{Ker}(f^3) \subseteq \cdots$ of submodules of M).