Corollary 6.10. Assume that R is commutative, unital and a PID. Let I be a nontrivial ideal in R. Then I is prime if and only if I is maximal.

Proof. If I is maximal, then I is prime by Example 6.8(3). Assume $I=(a)$ is prime. Let $J=(b)$ be an ideal in R with $I\subseteq J$ and $I\neq J$ and it is enough to show that $J=R$. Note that $I\neq(b)$ implies $b\notin I$. Since $a\in I\subseteq J=(b)$, there exists $x\in R$ with $a=bx$. Then $bx\in I$ and I is prime so that $b\in I$ or $x\in I$ by Theorem 6.9. Since $b\notin I$, we obtain $x\in I=(a)$. Then there exists $y\in R$ such that $x=ay$. Hence
$$a=bx=bay$$
and so $by=1$ (since R is an integral domain). Then $1=by\in(b)=J$ implies $J=R$. $\square$

Example 6.11. Assume R is commutative, unital and that every ideal in R is prime. Then we claim that R is a field. Indeed, if $ab=0$ in R, then $(a)(b)=(0)$ and so $(a)\subseteq(0)$ or $(b)\subseteq 0$ which gives $a=0$ or $b=0$. Hence R is an integral domain. Now let $a\in R\setminus\{0\}$. Then $a\cdot a\in(a^2)$ and since (a^2) is prime, we obtain that $a\in(a^2)$ by Theorem 6.9. Hence $a=ra^2$ for some $r\in R$ and so $ra=1$. This shows that a is invertible and so R is a field.