3. Ideals (Chapter 10.1)

R - ring

Definition 3.1. Let $(I,+)$ be a subgroup of $(R,+)$, that is
(i) $I \neq \emptyset$, and
(ii) $a,b \in I \Rightarrow a-b \in I$.
Then I is called a right ideal of R if
(iii) $a \in I, r \in R \Rightarrow ar \in I$
holds, and I is called a left ideal of R if
(iv) $a \in I, r \in R \Rightarrow ra \in I$
holds. If I is both a right and left ideal, that is if (i)–(iv) all hold, then we say that I is a two-sided ideal of R or simply an ideal of R.

Remark 3.2. (1) A left or right ideal is a subring.
(2) If R is commutative, then the notions of left ideal, right ideal and ideal coincide.
(3) $\{0\}$ and R are always ideals called trivial ideals.

Example 3.3. (1) Let I be a subring of $\mathbb{Z}$. Then $I \neq \emptyset$ and $a-b \in I$ for all $a,b \in I$ by Proposition 2.3. Moreover, if $a \in I$ and $r \in \mathbb{Z}$, then

$$ar = \begin{cases} \underbrace{a + \dots + a}_{r \text{ times}}, & r > 0, \\ 0, & r = 0, \\ \underbrace{-a - \dots - a}_{r \text{ times}}, & r < 0, \end{cases}$$

and so $ar \in I$. Since $\mathbb{Z}$ is commutative, it follows that I is an ideal. Hence subrings and ideals of $\mathbb{Z}$ coincide.

(2) Assume R is a unital ring and I is a right ideal of R. Assume that a unit $a \in R$ belongs to I. Then for every $r \in R$ we have that

$$r = a(a^{-1}r) \in I$$

where $a \in I$, $a^{-1}r \in R$. Hence $I = R$. The same holds for left and two-sided ideals of R which contain a unit. In particular, if R is a division ring, then the only ideals of R are the trivial ideals $\{0\}$ and R.

(3) Let F be a field and let $R = M_n(F)$. For $1 \le i \le n$ let A_i respectiely B_i be the subset of R consisting of all matrices with zeroes everywhere except possibly the i-th row respectively column. Then A_i is a right ideal in R and B_i is a left ideal in R.

Let $I \subseteq R$ be an ideal. We denote by $M_n(I)$ the set of all $n \times n$ matrices with entries in I.

<u>Theorem 3.4.</u> (1) If I is an ideal in R, then $M_n(I)$ is an ideal in $M_n(R)$

(2) If R is unital, then every ideal in $M_n(R)$ is of the form $M_n(I)$ for some ideal I in R.

(3) If R is a division ring, then $M_n(R)$ has no nontrivial ideals.

<u>Proof</u> (1) Exercise.

(2) Let $J \subseteq M_n(R)$ be an ideal. Set

$$I := \{a_{11} \mid A = (a_{ij}) \in J\}.$$

It is easy to see that I is an ideal in R. We claim that $J = M_n(I)$. Let $A \in M_n(R)$. Then

$$\underset{e_{ij}}{\underset{\shortparallel}{\begin{array}{cc} & \text{j-th column} \\ \text{i-th row} & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}}} \underset{A}{\underset{\shortparallel}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}} \underset{e_{kl}}{\begin{array}{cc} \text{l-th column} & \\ \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} & \text{k-th row} \end{array}}$$

$$= \begin{array}{cc} & \text{l-th column} \\ \text{i-th row} & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & a_{jk} & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array} = a_{jk} e_{il}.$$

Hence $a_{jk} e_{il} = e_{ij} A e_{kl}$ $\forall A \in M_n(R)$. Now we show $J \subseteq M_n(I)$. Let $A \in J$ and we need to show that $a_{ij} \in I$ for all $1 \leq i,j \leq n$. We have

$$\begin{pmatrix} a_{ij} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{pmatrix} = a_{ij} e_{11} = e_{1i} A e_{j1} \in J$$

and so $a_{ij} \in I$. Hence $A \in M_n(I)$ and so $J \subseteq M_n(I)$. Now let $B \in M_n(I)$. Then $b_{ij} \in I$ and so there exists $C^{ij} \in J$ with $c^{ij}_{11} = b_{ij}$. Then

$$b_{ij} e_{ij} = c^{ij}_{11} e_{ij} = e_{i1} C^{ij} e_{1j} \in J$$

and so

$$B = \sum_{1 \leq i,j \leq n} b_{ij} e_{ij} = \sum_{1 \leq i,j \leq n} e_{i1} C^{ij} e_{1j} \in J$$

which shows that $M_n(I) \subseteq J$.

(3) Let J be an ideal of $M_n(R)$. By (2) we have that $J = M_n(I)$ for some ideal I of R. Since R is a division ring, $I = \{0\}$ or $I = R$ by Example 3.3(2) and so $J = \{0\}$ or $J = M_n(R)$. $\square$