Proposition 2.21. The following are equivalent.
(a) R is a division ring.
(b) The equation $ax=b$ with $a,b\in R$ and $a\neq 0$ has a solution.

Proof (a)$\Rightarrow$(b): Since R is a division ring and $a\neq 0$, there exists $a^{-1}\in R$. Then $x=a^{-1}b$ is a solution.
(b)$\Rightarrow$(a): For every $a\in R\setminus\{0\}$ let e_a be the solution to $ax=a$. We claim that R is an integral domain. Assume to a contradiction that $ab=0$ for some $a,b\in R\setminus\{0\}$. Let y be such that $by=e_a$. Then $0=0y=aby=ae_a=a$, contradicting $a\neq 0$ and $b\neq 0$. Hence R is an integral domain. Hence if

$$ae_a=a=ae_a',$$

then $e_a=e_a'$ and so e_a is unique for each $a\in R\setminus\{0\}$. Then

$$ae_a^2=(ae_a)e_a=ae_a$$

gives $e_a^2=e_a$, and so e_a is idempotent. Hence for every $b\in R$ we have

$$(be_a-b)e_a=be_a^2-be_a=be_a-be_a=0$$

and since $e_a\neq 0$, we conclude $be_a=b$. By Lemma 2.17(2) we conclude that $e_a=1$ and R is unital. Hence for every $a\in R\setminus\{0\}$ there exists a solution to $ax=1$ and so every nonzero element in R has a right inverse. By the cancellation property this right inverse is unique. By Lemma 2.19 we conclude that every nonzero element of R has a two-sided inverse and hence R is a division ring. $\square$

Proposition 2.22. Assume that $r^3=r$ for all $r\in R$.
(1) If $e^2=e\in R$ is an idempotent, then $e\in Z(R)$.
(2) R is commutative.

Proof. (1) We need to show that $er = re$ for all $r \in R$. We have

$$e(r-er) = er - er = 0$$

and so

$$(r-er)e = \big((r-er)e\big)^3 = (r-er)\underbrace{e(r-er)}_{0}e(r-er)e = 0$$

$\Rightarrow \quad re = ere$

Similarly one sees that $er = ere$ and so $e \in Z(R)$.

(2) Let $r \in R$. Then $(r^2)^2 = r^4 = r^3 r = rr = r^2$ and so $r^2 \in Z(R)$ by (1). In particular $(r+r^2)^2 \in Z(R)$ for all $r \in R$. Then

$$r + r^2 = (r+r^2)^3 = (r+r^2)(r+r^2)^2 = (r+r^2)(r^2+2r^3+r^4)$$
$$= (r+r^2)(r^2+2r+r^2) = 2(r+r^2)^2 = \underset{\in Z(R)}{(r+r^2)} + \underset{\in Z(R)}{(r+r^2)}$$

and so $r + r^2 \in Z(R)$ $\forall\, r \in R$. But then

$$r = \underset{\in Z(R)}{(r+r^2)} - \underset{\in Z(R)}{r^2}$$

and so $r \in Z(R)$ $\forall\, r \in R$. Hence $R = Z(R)$ is commutative. □

Example 2.23 (1) The set

$$R[[X]] = \{(a_0, a_1, a_2, \ldots) \mid a_i \in R\}$$

with addition given by

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots)$$

and multiplication given by

$$(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (c_0, c_1, \ldots)$$

where $c_i = \sum_{j+k=i} a_j b_k$ is a ring called the ring of formal power series over R.

(2) The set

$$R\langle X\rangle = \{(\ldots, 0, a_{-k}, a_{-k+1}, \ldots, a_0, a_1, a_2, \ldots) \mid a_i \in R\}$$

with addition and multiplication as in $R[[X]]$ is a ring called the ring of formal Laurent series over R.

(3) Let G be a group and assume that R is unital. The set
$$R[G]=\{f:G\to R \mid f(a)=0 \text{ for all but finitely many } a\in G\}$$
with addition and zero element given by
$$(f+g)(a):=f(a)+g(a),$$
$$0(a):=0,$$
and multiplication and identity element given by
$$(fg)(a):=\sum_{bc=a} f(b)g(c)$$
$$1(a)=\begin{cases}1, & \text{if } a=e_G,\\ 0, & \text{otherwise},\end{cases}$$
is a ring called the <u>group ring</u> of G over R. If R is a field, then $R[G]$ is an R-algebra called a <u>group algebra</u>.

(4) A <u>quiver</u> $Q=(Q_0, Q_1, s, t)$ is a directed graph where
$$Q_0 = \text{set of } \underline{\text{vertices}},$$
$$Q_1 = \text{set of } \underline{\text{arrows}},$$
$$s,t: Q_1 \longrightarrow Q_0 \text{ are maps},$$
where if $i \xrightarrow{\alpha} j$ is an arrow in Q, the <u>source</u> map s gives $s(\alpha)=i$ and the <u>target</u> map t gives $t(\alpha)=j$. A <u>path</u> in Q is a sequence of arrows
$$\alpha_0\alpha_1\cdots\alpha_k$$
where $t(\alpha_i)=s(\alpha_{i-1})$ for $i=1,\dots,k$. We also define a <u>trivial path</u> e_v for each vertex $v\in Q_0$. If K is a field, then the <u>path algebra</u> KQ has
- as a K-vector space, a K-basis is given by all paths in Q,
- as a ring, a multiplication given by concatenation of paths when possible and 0 otherwise and extended bilinearly.

This makes KQ a K-algebra. A concrete example is given by

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

Then

$$KQ = \{a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 \alpha + a_5 \beta + a_6 \beta\alpha \mid a_i \in K\}.$$

In this case

$$KQ \cong U_3(K) = \left\{ \begin{pmatrix} a_1 & a_5 & a_6 \\ 0 & a_2 & a_4 \\ 0 & 0 & a_3 \end{pmatrix} \middle| a_i \in K \right\}.$$